5 References and Appendices

5.1 Appendix 1

This appendix contains proofs of the lemmas from Chapter 2.

Lemma 1:

The optimal capacity choice, y_I*, for a central decision-maker is given by,

$$y_I^* = F_X^{-1}\left(\frac{m_I - c_I}{m_I - v_I}\right) = F_X^{-1}\left(\frac{r - p_M - p_S - c_M - c_S}{r - p_M - p_S - v_M - v_S}\right)$$

Proof:

The total channel profit as a function of y_I, $\Pi(y_I)$ is given by,

$$\Pi(y_I) = -c_I y_I + m_I \int_a^{y_I} x f_X(x)dx + m_I y_I [1 - F_X(y_I)] + v_I \int_a^{y_I} [y_I - x] f_X(x)dx$$

Taking the derivative with respect to y_I, $\frac{d\Pi(y_I)}{dy_I} = -c_I + m_I - (m_I - v_I)F_X(y_I)$. The second derivative with respect to y_I, is $\frac{d^2\Pi(y_I)}{dy_I{}^2} = -(m_I - v_I) f_X(y_I)$

The retail price is restricted to $r>p_M+p_S+c_M+c_S$. The salvage values are also strictly less than the capacity costs. Therefore m_I-v_I>0. The density function is non-negative, therefore,

$$\frac{d^2\Pi(y_I)}{dy_I{}^2} \leq 0$$

and $\Pi(y_I)$ is a concave function of y_I. Therefore, the first order condition is sufficient for y_I* to be the profit maximizing capacity. So,

$$\left.\frac{d\Pi(y_I)}{dy_I}\right|_{y_I = y_I^*} = 0 \quad \Rightarrow y_I^* = F_X^{-1}\left(\frac{m_I - c_I}{m_I - v_I}\right)$$

Lemma 2:

For a given wholesale price, w, the optimal capacity choice, $y_M(w)$, for the manufacturer, assuming the supplier has infinite capacity, is given by,

$$y_M(w) = F_X^{-1}\left(\frac{m_M - c_M}{m_M - v_M}\right) = F_X^{-1}\left(\frac{r - w - p_M - c_M}{r - w - p_M - v_M}\right)$$

Proof:

As for Lemma 1 above, but replacing capacity, salvage and unit margin parameters as appropriate.

Lemma 3:

For a given wholesale price, w, the optimal capacity choice, $y_S(w)$, for the supplier, assuming the manufacturer has infinite capacity, is given by,

$$y_S(w) = F_X^{-1}\left(\frac{m_S - c_S}{m_S - v_S}\right) = F_X^{-1}\left(\frac{w - p_S - c_S}{w - p_S - v_S}\right)$$

Proof:

As for Lemma 1 above, but replacing capacity, salvage and unit margin parameters as appropriate.

Lemma 4:

(i) $y_M(w)$ is strictly decreasing in w

(ii) $y_S(w)$ is strictly increasing in w

(Note that $F_X(x)$ is assumed to be continuous and differentiable)

Proof:

(i)

$$\frac{dy_M(w)}{dw} = \left(\frac{1}{f_X(y_M(w))}\right)\left(\frac{v_M - c_M}{(r - w - p_M - v_M)^2}\right) < 0$$

(ii)

$$\frac{dy_S(w)}{dw} = \left(\frac{1}{f_X(y_S(w))}\right)\left(\frac{c_S - v_S}{(w - p_S - v_S)^2}\right) > 0$$

Lemma 5:

(i) There is a unique wholesale price, w, such that $y_M(w)=y_S(w)$

(ii) This unique wholesale price, w_{crit}, is given by,

$$w_{crit} = \frac{(r-p_M)(c_S - v_S) + p_S(c_M - v_M) + c_M v_S - c_S v_M}{c_S - v_S + c_M - v_M}$$

(iii) At this wholesale price, w_{crit}, $y_M(w_{crit})=y_S(w_{crit})=y_I*$

(iv) If $w=w_{crit}$, then the total channel profits equals the channel profits obtained by a central decision-maker.

Proof:

(i) and (ii)From Lemma 4, $y_M(w)$ is strictly decreasing in w and $y_S(w)$ is strictly increasing in w. Therefore $y_M(w)$ and $y_S(w)$ can cross each other at most once.

$$y_M(w) = F_X^{-1}\left(\frac{m_M - c_M}{m_M - v_M}\right) \qquad y_S(w) = F_X^{-1}\left(\frac{m_S - c_S}{m_S - v_S}\right)$$

So, $y_M(w)=y_S(w)$ iff,

$$\left(\frac{m_M - c_M}{m_M - v_M}\right) = \left(\frac{m_S - c_S}{m_S - v_S}\right)$$

$m_M=r\text{-}w\text{-}p_M$ and $m_S=w\text{-}p_S$. So, $y_M(w)=y_S(w)$ only if,

$$w = \frac{(r-p_M)(c_S - v_S) + p_S(c_M - v_M) + c_M v_S - c_S v_M}{(c_S - v_S + c_M - v_M)}$$

(iii) $y_I*=y_S(w)$ iff,

$$\left(\frac{m_I - c_I}{m_I - v_I}\right) = \left(\frac{m_S - c_S}{m_S - v_S}\right)$$

$m_I=r\text{-}p_M\text{-}p_S$, $c_I=c_M+c_S$, $v_I=v_M+v_S$ and So, $y_I*=y_S(w)$ iff,

$$w = \frac{(r-p_M)(c_S - v_S) + p_S(c_M - v_M) + c_M v_S - c_S v_M}{(c_S - v_S + c_M - v_M)}$$

which is the same condition for $y_M(w)=y_S(w)$ so at w_{crit}, $y_M(w_{crit})=y_S(w_{crit})=y_I*$

(iv) The total channel profits depend only on the supplier and manufacturer capacity. Since, $y_M(w_{crit})=y_S(w_{crit})=y_I*$, the capacities are the same as those chosen by a central decision-maker and therefore the total channel profits are the same.

Lemma 6:

The necessary and sufficient condition on the demand distribution, $F_X(x)$, for $y_S(w)$ to be a concave function of w is given by,

$$\frac{f'_X(x)[1-F_X(x)]}{[f_X(x)]^2} \geq -2 \quad \forall\, x \in [a,b]$$

For strict concavity, the inequality needs to be strict. This condition is referred to as the (strict) concavity condition in the rest of the proofs. Note that this condition is equivalent to requiring that $g'(x) \geq 0\ \forall$ x, where,

$$g(x) = \frac{f_X(x)}{[1-F_X(x)]^2}$$

Proof:

$$y_S(w) = F_X^{-1}\left(\frac{m_S - c_S}{m_S - v_S}\right) = F_X^{-1}\left(\frac{w - p_S - c_S}{w - p_S - v_S}\right)$$

Therefore,

$$\frac{d^2 y_S(w)}{dw^2} = \left(\frac{1}{f_X(y_S(w))}\right)\left(\frac{(1-F_X[y_S(w)])^3}{(c_S - v_S)^2}\right)\left(-2-\left(\frac{f'_X(y_S(w))(1-F_X[y_S(w)])}{[f_X(y_S(w))]^2}\right)\right)$$

The first bracketed term is > 0. Now, because $v_S < c_S$, i.e. the salvage value is strictly less than capacity cost,

$$F_X[y_S(w)] = \left(\frac{w - p_S - c_S}{w - p_S - v_S}\right) < 1$$

Therefore, the second bracketed term is > 0. So, for

$$\frac{d^2 y_S(w)}{dw^2} \leq 0$$

the third bracketed term needs to be ≤ 0, and for strict concavity this term needs to be < 0. Indeed this condition is sufficient. So, the necessary and sufficient condition for $y_S(w)$ to be a concave function of w is then given by,

$$\frac{f'_X(y_S(w))[1-F_X(y_S(w))]}{[f_X(y_S(w))]^2} \geq -2$$

Since $y_S(w)$ is strictly increasing in w and since for every $x \in [a,b]$ there is wholesale price w such that $y_S(w)=x$, then this condition is satisfied iff

$$\frac{f_X'(x)[1-F_X(x)]}{\left[f_X(x)\right]^2} \geq -2 \quad \forall\, x \in [a,b]$$

Lemma 7:

(i) If $F_X(x)$ is an increasing failure rate distribution (IFR), then,

$$\frac{f_X'(x)[1-F_X(x)]}{\left[f_X(x)\right]^2} > 1 \quad \forall\, x$$

and (ii) the strict concavity condition is satisfied so $y_S(w)$ is a strictly concave function.

Proof:

(i)The failure rate function, $h(x)$, for a distribution $F_X(x)$, is defined as,

$$h(x) = \frac{f_X(x)}{1-F_X(x)}$$

A distribution is said to have an increasing failure rate function (IFR) if $h'(x) > 0 \quad \forall\, x$

$$h'(x) = \frac{[1-F_X(x)]\,f_X'(x) - [f_X(x)]^2}{[1-F_X(x)]^2}$$

If $F_X(x)$ is an IFR distribution, then

$$h'(x) > 0 \quad \forall\, x \Rightarrow \frac{[1-F_X(x)]\,f_X'(x) - [f_X(x)]^2}{[1-F_X(x)]^2} > 0 \quad \forall\, x$$

$$\Rightarrow \frac{[1-F_X(x)]\,f_X'(x)}{[f_X(x)]^2} > 1 \quad \forall\, x$$

(ii) Follows directly from part (i) and Lemma 6 above

Lemma 8:

In *Game A*, where the wholesale price is exogenous and the manufacturer is the Stackleberg leader, the manufacturer and supplier choose their capacities such that $y_M^* = y_S^* = min\{y_S(w), y_M(w)\}$, where $y_M(w)$ and $y_S(w)$ are given by Lemma 2 and Lemma 3 respectively.

Proof:

The supplier never chooses a capacity larger than the manufacturer's capacity choice, as its total sales are limited by the manufacturer's capacity. From Lemma 3, the supplier's profit,

$\Pi_S(y)$, is concave with the maximum achieved at $y_S(w)$. For $y<y_S(w)$, $\Pi_S(y)$ is increasing in y. Therefore, if the manufacturer announces a capacity of y, the supplier chooses a capacity of $min\{y,y_S(w)\}$. Likewise, the manufacturer never announces a capacity larger than the supplier would choose, as its total sales are limited by the supplier's capacity choice. From Lemma 2, the manufacturer's profit, $\Pi_M(y)$, is concave with the maximum achieved at $y_M(w)$. For $y<y_M(w)$, $\Pi_M(y)$ is increasing in y. Therefore, the manufacturer chooses its capacity, y_M*=min$\{y_S(w),y_M(w)\}$. The supplier chooses a capacity of $min\{y_M*,y_S(w)\}=min\{y_S(w),y_M(w)\}$.

Lemma 9:

In *Game B*, where the manufacturer chooses the wholesale price, the manufacturer never chooses a wholesale price, w, such that $w>w_{crit}$.

Proof:

Let the manufacturer choose a wholesale price, $w*>w_{\text{crit}}$. The channel capacity is then given by min$\{y_S(w*),y_M(w*)\}$. For $w>w_{crit}$, $y_S(w)>y_M(w)$, so the channel capacity is $y_M(w*)$. However, there exists a $w**<w_{crit}$ such that $y_S(w**)=y_M(w*)$. [For every $x\in$ [a,b] there is a wholesale price w_S such that that $y_S(w_S)=x$. Likewise there is a unique wholesale price w_M such that $y_M(w_M)=x$. $y_S(w)$ is strictly increasing in w, $y_M(w)$ is strictly decreasing in w and there is a unique wholesale price w_{crit} for which $y_S(w_{crit})=y_M(w_{crit})$.]

The manufacturer's expected total sales depend only on the channel capacity. Thus the manufacturer's expected sales revenue is the same for w* and w**, as is the manufacturer's capacity cost and expected salvage revenue. The cost per unit sold is strictly less if w** is chosen. Therefore the manufacturer's profit is strictly greater at w** and so the manufacturer never chooses a $w>w_{crit}$.

Therefore, the manufacturer's wholesale price choice can be restricted to $w\leq w_{crit}$. For the supplier to invest in capacity, the wholesale price, w, must be larger than the sum of unit capacity cost and unit marginal cost ($w\geq p_S+c_S$). So the manufacturer's optimal wholesale price falls within $p_S+c_S<w\leq w_{crit.}$

Lemma 10:

If $F_X(x)$ satisfies the concavity condition, then the manufacturer's profit, $\Pi_M(w)$, restricted to $p_S+c_S<w\leq w_{crit}$, is a strictly concave function.

Proof:

In this range the manufacturer's profit, $\Pi_M(w)$, is given by,

$$\Pi_M(w) = -c_M y_S(w) + (r - p_M - w)\int_a^{y_S(w)} x f_X(x)dx + (r - p_M - w) y_S(w)[1 - F_X(y_S(w))]$$
$$+ v_M \int_a^{y_S(w)} [y_S(w) - x] f_X(x)dx$$

Taking the derivative with respect to *w*,

$$\frac{d\Pi_M(w)}{dw} = \left(\frac{dy_S(w)}{dw}\right)\left((r - p_M - w - c_M) - (r - p_M - w - v_M)F_X(y_S(w))\right)$$
$$- \int_a^{y_S(w)} x f_X(x)dx - y_S(w)[1 - F_X(y_S(w))]$$

Taking the second derivative,

$$\frac{d^2\Pi_M(w)}{dw^2} = \left(\frac{d^2 y_S(w)}{dw^2}\right)\left((r - p_M - w - c_M) - (r - p_M - w - v_M)F_X(y_S(w))]\right)$$
$$-2\left(\frac{dy_S(w)}{dw}\right)[1 - F_X(y_S(w)] - (r - p_M - w - v_M)\, f_X(y_S(w))\left(\frac{dy_S(w)}{dw}\right)^2$$

Now,

(i) From Lemma 4(ii), $\frac{dy_S(w)}{dw} > 0$

(ii) As $F_X(x)$ satisfies the concavity condition, then $\frac{d^2 y_S(w)}{dw^2} \le 0$

(iii) $r>w+p_M+c_M$ in the allowable range for *w*,

(iv) $c_M>v_M$

(v) In the specified range for *w*,

$$F_X[y_S(w)] = \left(\frac{w - p_S - c_S}{w - p_S - v_S}\right) \le F_X[y_M(w)] = \left(\frac{r - w - p_M - c_M}{r - w - p_M - v_M}\right) < 1$$

Using (i)-(v), $\frac{d^2\Pi_M(w)}{dw^2} < 0$

Lemma 11:

If $F_X(x)$ satisfies the concavity condition (eqn. (2)) given in Lemma 6, then the optimal w for Game B is given by the first order condition,

$$\frac{d\Pi_M(w)}{dw} = \left(\frac{dy_S(w)}{dw}\right)\left((r - p_M - w - c_M) - (r - p_M - w - v_M)F_X(y_S(w))\right) - \int_a^{y_S(w)} xf_X(x)dx - y_S(w)[1 - F_X(y_S(w))] = 0$$

and the optimal w is strictly greater than p_S+c_S and strictly less than w_{crit}.

Proof:

From Lemma 9, the manufacturer's optimal wholesale price lies within $p_S+c_S<w^*\leq w_{crit}$. From Lemma 10, $\Pi_M(w)$ is a strictly concave function so the first order condition is sufficient for optimality as long as the wholesale price, w^*, that satisfies the condition, lies in the interior of the range, $p_S+c_S<w\leq w_{\mathrm{crit}}$. I will show that the optimal wholesale price satisfies $p_S+c_S<w^*<w_{crit}$.

If $w\leq p_S+c_S$, then the supplier does not invest in any capacity and the manufacturer's profit is be zero. At w_{crit} the manufacturer's profit is strictly positive. $\Pi_M(w)$ is a strictly concave function in the range $p_S+c_S\leq w\leq w_{crit}$, so we must have

$$\left.\frac{d\Pi_M(w)}{dw}\right|_{w=p_S+c_S} > 0$$

At $w=w_{crit}$,

…. (L11.1)

$$\left.\frac{d\Pi_M(w)}{dw}\right|_{w=w_{crit}} = -\int_a^{y_S(w_{crit})} xf_X(x)dx - y_S(w_{crit})[1 - F_X(y_S(w_{crit}))] < 0$$

…. (L11.2)

$\Pi_M(w)$ is concave function. From (L11.2) the profit is decreasing at $w=w_{crit}$, so the first order condition must be satisfied for $w<w_{crit}$. From (L11.1) the profit is increasing at $w=p_S+c_S$. Therefore, the w^* that satisfies the first order condition must satisfy $p_S+c_S<w^*<w_{crit}$.

Lemma 12:

If $F_X(x)$ satisfies the concavity condition, then in *Game B*, the total channel profits, Π_D, is strictly less than the total channel profits obtained by a central decision-maker, Π_C.

Proof:

$\Pi_d=\Pi_c$ only if the channel capacity chosen is the same as that chosen by a central decision-maker. This only happens if the manufacturer chooses the wholesale price, w, such that $w=w_{crit}$. From Lemma 11, this never occurs. Therefore, the total channel profits is strictly less than those obtained by a central decision-maker.

Lemma 13:

For any allowable price schedule (w,Δ), $p_S+c_S\leq w+\Delta\leq r-c_M-p_M$ and $\Delta\geq 0$, then (i) the optimal capacity choice, $y_S(w,\Delta)$, for the supplier, assuming the manufacturer has infinite capacity, is given by,

$$y_S(w,\Delta)=F_X^{-1}\left(\frac{m_S+\Delta-c_S}{m_S+\Delta-v_S}\right)=F_X^{-1}\left(\frac{w+\Delta-p_S-c_S}{w+\Delta-p_S-v_S}\right)\geq y_S(w)$$

and (ii) the optimal capacity choice, $y_M(w,\Delta)$, for the manufacturer, assuming the supplier has infinite capacity, is given by,

$$y_M(w,\Delta)=F_X^{-1}\left(\frac{m_M-\Delta-c_M}{m_M-\Delta-v_M}\right)=F_X^{-1}\left(\frac{r-w-\Delta-p_M-c_M}{r-w-\Delta-p_M-v_M}\right)\leq y_M(w)$$

Proof:

(i) Let $\Pi_S(y,w,\Delta)$ denote the supplier's profit as a function of y for a price schedule of (w,Δ).If $\Delta=0$, then the supplier invests in capacity, $y_S(w)$. However, $\Delta>0$, so when determining the optimal capacity, $y_S(w,\Delta)$, we only need to look at $y\geq y_S(w)$. In this range,

$$\begin{aligned}\Pi_S(y,w,\Delta)=-c_S y+m_S\int_a^{y_S(w)}xf_X(x)dx+m_S y_S(w)[1-F_X(y_S(w)]\\+(m_S+\Delta)\int_{y_S(w)}^{y}(x-y_S(w))f_X(x)dx+(m_S+\Delta)(y-y_S(w))][1-F_X(y)]\\+v_S\int_a^{y}[y-x]f_X(x)dx\end{aligned}$$

Taking the first derivative with respect to y, $\frac{d\Pi_S(y,w,\Delta)}{dy} = m_S + \Delta - c_S - (m_S + \Delta - v_S)F_X(y)$

Taking the second derivative,

$$\frac{d^2\Pi_S(y,w,\Delta)}{dy^2} = -(m_S + \Delta - v_S)f_X(y) \le 0$$

So, $\Pi_S(y,w,\Delta)$ is a concave function of y and the first order condition is sufficient for optimality. Thus,

$$F_X(y_S(w,\Delta)) = \left(\frac{m_S + \Delta - c_S}{m_S + \Delta - v_S}\right) = \left(\frac{w + \Delta - p_S - c_S}{w + \Delta - p_S - v_S}\right)$$

(ii) Proof follows similarly to (i) but the quantity premium, Δ, is subtracted from the unit margin.

Important Note for Proofs of Lemmas 14, 15 and 16:

(i) $y_S(w,\Delta)=y_S(w')$ and $y_M(w,\Delta)=y_M(w')$ where $w'=w+\Delta$.

(ii) The first and second derivatives of $y_S(w,\Delta)$ with respect to either w or Δ is the same as the derivatives of $y_S(w')$ with respect to w'.

(iii) The first and second derivatives of $y_M(w,\Delta)$ with respect to either w or Δ is the same as the derivatives of $y_M(w')$ with respect to <u>w</u>'.

Given (i),(ii) and (iii), Lemmas 14, 15 and 16 follow directly from the proofs of Lemmas 4, 5 and 6. Fully worked proofs, independent of Lemmas 4, 5 and 6 are available.

Lemma 14:

(i) $y_M(w,\Delta)$ is strictly decreasing in both w and Δ.

(ii) $y_S(w,\Delta)$ is strictly increasing in both w and Δ.

(Note that $F_X(x)$ is assumed to be continuous and differentiable)

Proof:

Follows from Lemma 4 and above note.

Lemma 15:

For a given wholesale price $w \le w_{crit}$ there is a unique quantity premium $\Delta_{crit,}$ given by

$$w+\Delta_{crit}=\frac{(r-p_M)(c_S-v_S)+p_S(c_M-v_M)+c_M v_S-c_S v_M}{c_S-v_S+c_M-v_M}=w_{crit}\,,$$

such that $y_M(w,\Delta_{crit})=y_S(w,\Delta_{crit})=y_I*$.

Proof:

Follows from Lemma 5 and above note.

Lemma 16:

(i) $y_S(w,\Delta)$ is a concave function of both w and Δ iff the concavity condition given by equation (2) holds for $F_X(x)$.

(ii) If $F_X(x)$ is an IFR distribution then equation (2) is satisfied so $y_S(w,\Delta)$ is a strictly concave function of both Δ and w.

Proof:

(i) From Lemma 6 $\frac{\partial^2 y_S(w,\Delta)}{\partial w^2}\le 0$ if and only if the concavity condition holds. From the above note, $\frac{\partial^2 y_S(w,\Delta)}{\partial w\partial\Delta}=\frac{\partial^2 y_S(w,\Delta)}{\partial\Delta^2}=\frac{\partial^2 y_S(w,\Delta)}{\partial w^2}$. Therefore, $\frac{\partial^2 y_S(w,\Delta)}{\partial w^2}\le 0$, $\frac{\partial^2 y_S(w,\Delta)}{\partial\Delta^2}\le 0$ and $\left(\frac{\partial^2 y_S(w,\Delta)}{\partial w^2}\right)\left(\frac{\partial^2 y_S(w,\Delta)}{\partial\Delta^2}\right)-\left(\frac{\partial^2 y_S(w,\Delta)}{\partial w\partial\Delta}\right)^2\le 0$. So $y_S(w,\Delta)$ is a concave function of w and Δ.

(ii) Follows directly from Lemma 7.

Lemma 17:

For Game C the manufacturer never chooses a quantity premium, Δ, such that $\Delta>\Delta_{crit}$.

Proof:

Similar to proof of Lemma 9 but adapting for Δ instead of w. Therefore, the manufacturer's wholesale price choice can be restricted to $0\le\Delta\le\Delta_{crit}$.

Lemma 18:

For Game C, if $F_X(x)$ satisfies the concavity condition given by equation (2), then the manufacturer's profit, $\Pi_M(w,\Delta)$, restricted to $0 \le \Delta \le \Delta_{crit}$, is a strictly concave function of Δ.

Proof:

The manufacturer's profit, $\Pi_M(w,\Delta)$, is given by,

$$\begin{aligned}\Pi_M(w,\Delta) = -c_M y_S(w,\Delta) + m_M \int_a^{y_S(w)} x f_X(x)dx + m_M y_S(w)[1 - F_X(y_S(w)]\\ + (m_M - \Delta) \int_{y_S(w)}^{y_S(w,\Delta)} (x - y_S(w)) f_X(x)dx\\ + (m_M - \Delta)(y_S(w,\Delta) - y_S(w))][1 - F_X(y_S(w,\Delta))]\\ + v_M \int_a^{y_S(w,\Delta)} [y_S(w,\Delta) - x] f_X(x)dx\end{aligned}$$

Taking the derivative with respect to Δ,

$$\begin{aligned}\frac{d\Pi_M(w,\Delta)}{d\Delta} = \left(\frac{dy_S(w,\Delta)}{d\Delta}\right)\left[(m_M - \Delta - c_M) - (m_M - \Delta - v_M)F_X(y_S(w,\Delta))\right]\\ - \int_{y_S(w)}^{y_S(w,\Delta)} (x - y_S(w)) f_X(x)dx - (y_S(w,\Delta) - y_S(w))[1 - F_X(y_S(w,\Delta))]\end{aligned}$$

Taking the second derivative,

$$\begin{aligned}\frac{d^2\Pi_M(w,\Delta)}{d\Delta^2} = \left(\frac{d^2 y_S(w,\Delta)}{d\Delta^2}\right)\left[(m_M - \Delta - c_M) - (m_M - \Delta - v_M)F_X(y_S(w,\Delta))\right]\\ -2\left(\frac{dy_S(w,\Delta)}{d\Delta}\right)\left[1 - F_X(y_S(w,\Delta)\right] - (m_M - \Delta - v_M) f_X(y_S(w,\Delta))\left(\frac{dy_S(w,\Delta)}{d\Delta}\right)^2\end{aligned}$$

Now,

(i) From Lemma 14(ii), $\frac{dy_S(w,\Delta)}{d\Delta} > 0$

(ii) As $F_X(x)$ satisfies concavity, then from Lemma 16(i), $\frac{d^2 y_S(w,\Delta)}{d\Delta^2} \le 0$

(iii) $m_M > \Delta + c_M$ in the allowable range for Δ,

(iv) $c_M > v_M$

(v) In the specified range for w,

$$F_X\left[y_S(w)\right]=\left(\frac{m_S+\Delta-c_S}{m_S+\Delta-v_S}\right)\leq F_X\left[y_M(w)\right]=\left(\frac{m_M-\Delta-c_M}{m_M-\Delta-v_M}\right)<1$$

Using (i)-(v), $\frac{d^2\Pi_M(w,\Delta)}{d\Delta^2}<0$

Lemma 19:

For Game C,

(i) the manufacturer chooses a positive quantity premium, i.e. $\Delta^*(w)>0$

(ii) the supplier's profit strictly increases with increasing Δ

(iii) $\Pi_T(w,\Delta^*(w))$ the expected total supply chain profit for the price schedule $(w,\Delta^*(w))$, is strictly greater than the expected total supply chain profit $\Pi_T(w)$ when no quantity premium is offered but the supply chain is not completely coordinated, i.e. $\Pi_T(w)<\Pi_T(w_{crit})$.

Proof:

(i) The derivative of $\Pi_M(w,\Delta)$ with respect to Δ is given in the proof of Lemma 18,

$$\frac{d\Pi_M(w,\Delta)}{d\Delta}=\left(\frac{dy_S(w,\Delta)}{d\Delta}\right)\left[(m_M-\Delta-c_M)-(m_M-\Delta-v_M)F_X(y_S(w,\Delta))\right]$$
$$-\int_{y_S(w)}^{y_S(w,\Delta)}(x-y_S(w))f_X(x)dx-(y_S(w,\Delta)-y_S(w))[1-F_X(y_S(w,\Delta))]$$

So, at $\Delta=0$,

$$\left.\frac{d\Pi_M(w,\Delta)}{d\Delta}\right|_{\Delta=0}=\left(\left.\frac{dy_S(w,\Delta)}{d\Delta}\right|_{\Delta=0}\right)\left[(m_M-c_M)-(m_M-v_M)F_X(y_S(w))\right]$$

From lemma 14(ii), the first bracketed term is > 0. The second bracketed term is also >0. Therefore,

$$\left.\frac{d\Pi_M(w,\Delta)}{d\Delta}\right|_{\Delta=0}>0$$

From Lemma 18 $\Pi_M(w,\Delta)$ is a strictly concave function of Δ. $\Pi_S(w,\Delta)$ is increasing at $\Delta=0$ and so the optimal Δ^* is strictly greater than 0.

(ii) For a fixed wholesale price, w, the supplier's profit as a function of Δ, $\Pi_S(w,\Delta)$, is given by,

$$\begin{aligned}\Pi_S(w,\Delta) = &-c_S y_S(w,\Delta) + m_S \int_a^{y_S(w)} x f_X(x)dx + m_S y_S(w)[1 - F_X(y_S(w)]\\ &+ (m_S + \Delta) \int_{y_S(w)}^{y_S(w,\Delta)} (x - y_S(w)) f_X(x)dx\\ &+ (m_S + \Delta)(y_S(w,\Delta) - y_S(w))][1 - F_X(y_S(w,\Delta))]\\ &+ v_S \int_a^{y_S(w,\Delta)} [y_S(w,\Delta) - x] f_X(x)dx\end{aligned}$$

Taking the derivative with respect to Δ,

$$\begin{aligned}\frac{d\Pi_S(w,\Delta)}{d\Delta} = &\left(\frac{dy_S(w,\Delta)}{d\Delta}\right)\left[(m_S + \Delta - c_S) - (m_S - \Delta - v_S)F_X(y_S(w,\Delta))\right]\\ &+ \int_{y_S(w)}^{y_S(w,\Delta)} (x - y_S(w)) f_X(x)dx + (y_S(w,\Delta) - y_S(w))[1 - F_X(y_S(w,\Delta))]\\ &> 0\end{aligned}$$

So for a fixed wholesale price $w<w_{crit}$, the supplier's expected profit increases if the manufacturer offers a positive quantity premium.

(iii) From (i) the manufacturer chooses a Δ>0 and its profit strictly increases. From (ii) the supplier's profit strictly increases. Therefore the total channel profits are greater than in Game A, i.e. $\Pi_T(w,\Delta^*(w))>\Pi_T(w)$. From (i) $0<\Delta^*(w)$. I will now show that $\Delta^*(w)<\Delta_{crit}$.

$$\begin{aligned}\left.\frac{d\Pi_M(w,\Delta)}{d\Delta}\right|_{\Delta=\Delta_{crit}} = &- \int_{y_S(w)}^{y_S(w,\Delta_{crit})} (x - y_S(w)) f_X(x)dx - (y_S(w,\Delta_{crit}) - y_S(w))[1 - F_X(y_S(w,\Delta_{crit}))]\\ &< 0\end{aligned}$$

$\Pi_M(w,\Delta)$, restricted to $0\leq\Delta\leq\Delta_{crit}$, is a concave function of Δ. Therefore, because

$$\left.\frac{d\Pi_M(w,\Delta)}{d\Delta}\right|_{\Delta=0} > 0 \qquad \left.\frac{d\Pi_M(w,\Delta)}{d\Delta}\right|_{\Delta=\Delta_{crit}} < 0$$

the optimal Δ for the manufacturer is given by the first order condition and $0<\Delta^*(w)<\Delta_{crit}$. Therefore the channel capacity is strictly less than that chosen by a central decision-maker and thus the total channel profit is strictly less then those obtained by a central decision-maker, i.e.

$\Pi_T(w,\Delta^*(w))<\Pi_C=\Pi_T(w_{crit})$, where the central decision-maker expected profit is the same as the total supply chain profit in Game A if $w=w_{crit}$ (Lemma 5).

Lemma 20:

(i) If $F_X(x)$ satisfies the concavity condition, then the manufacturer's profit $\Pi_M(w,\Delta)$, restricted to $p_S+c_S<w+\Delta<w_{crit}$ and $\Delta>0$, is a strictly concave function of w and Δ.

(ii) The first order conditions for w and Δ are necessary and sufficient for (w^*,Δ^*) to be optimal.

Proof:

(i) From Lemma 18,

$$\frac{\partial^2\Pi_M(y_S(w,\Delta))}{\partial\Delta^2}<0$$

Next, I will show that,

$$\frac{\partial^2\Pi_M(y_S(w,\Delta))}{\partial w^2}<0$$

$$\begin{aligned}\Pi_M(w,\Delta)=&-c_M y_S(w,\Delta)+m_M\int_a^{y_S(w)}xf_X(x)dx+m_M y_S(w)[1-F_X[y_S(w)]]\\&+(m_M-\Delta)\int_{y_S(w)}^{y_S(w,\Delta)}(x-y_S(w))f_X(x)dx\\&+(m_M-\Delta)[y_S(w,\Delta)-y_S(w)][1-F_X[y_S(w,\Delta)]]\\&+v_M\int_{y_S(w)}^{y_S(w,\Delta)}(y_S(w,\Delta)-x)f_X(x)dx\end{aligned}$$

Taking the partial derivative with respect to the wholesale price w,

$$\begin{aligned}\frac{\partial\Pi_M(w,\Delta)}{\partial w}=&+\Delta\left(\frac{dy_S(w)}{dw}\right)[1-F_X[y_S(w)]]-\int_a^{y_S(w,\Delta)}xf_X(x)dx-y_S(w,\Delta)[1-F_X[y_S(w,\Delta)]]\\&+\left(\frac{\partial y_S(w,\Delta)}{\partial w}\right)((m_M-\Delta-c_M)-(m_M-\Delta-v_M)F_X[y_S(w,\Delta)])\end{aligned}$$

Taking the second derivative with respect to w,

$$\frac{\partial^2 \Pi_M(w,\Delta)}{\partial w^2} = \Delta\left(\frac{d^2 y_S(w)}{dw^2}\right)[1 - F_X[y_S(w)]] - \Delta\left(\frac{dy_S(w)}{dw}\right)^2 f_X(y_S(w))$$

$$+\left(\frac{\partial^2 y_S(w,\Delta)}{\partial^2 w}\right)\left((m_M - \Delta - c_M) - (m_M - \Delta - v_M)F_X[y_S(w,\Delta)]\right)$$

$$-2\left(\frac{\partial y_S(w,\Delta)}{\partial w}\right)[1 - F_X[y_S(w,\Delta)]]$$

$$-(m_M - \Delta - v_M) f_X(y_S(w,\Delta))\left(\frac{\partial y_S(w,\Delta)}{\partial w}\right)^2$$

Now,

(i) $\Delta \geq 0$,

(ii) $\frac{\partial y_S(w,\Delta)}{\partial w} > 0$ from Lemma 14(ii)

(iii) As $F_X(x)$ satisfies the concavity condition, $\frac{d^2 y_S(w)}{dw^2} \leq 0$ and $\frac{\partial^2 y_S(w,\Delta)}{\partial w^2} \leq 0$

(iv) $m_M \geq \Delta + c_M$ is the allowable range for w

(v) $c_M > v_M$

(vi) In the specified range for w,

$$F_X[y_S(w)] = \left(\frac{m_S + \Delta - c_S}{m_S + \Delta - v_S}\right) \leq F_X[y_M(w)] = \left(\frac{m_M - \Delta - c_M}{m_M - \Delta - v_M}\right) < 1$$

Using (i)-(vi), $\frac{\partial^2 \Pi_M(w,\Delta)}{\partial w^2} < 0$

I will now use the following theorem (Theorem 2.13 from Avriel, Diewert, Schaible and Zang ,1988): Let f be a differentiable function on the open convex set $C \subset R^n$. It is concave if and only if for every two points $\mathbf{x}^1 \in C$, $\mathbf{x}^2 \in C$,

$$(\mathbf{x}^2 - \mathbf{x}^1)^T [\nabla f(\mathbf{x}^2) - \nabla f(\mathbf{x}^1)] \leq 0$$

It is strictly concave if and only if this inequality is strict for $\mathbf{x}^1 \neq \mathbf{x}^2$.

So, for this problem the condition is that for every (w^1, Δ^1) and (w^2, Δ^2) in the allowable region ($p_S + c_S < w + \Delta < w_{crit}$ and $\Delta > 0$),

$$(w^2 - w^1)\left[\left.\frac{\partial \Pi_M(w,\Delta)}{\partial w}\right|_{w=w^2} - \left.\frac{\partial \Pi_M(w,\Delta)}{\partial w}\right|_{w=w^1}\right] + (\Delta^2 - \Delta^1)\left[\left.\frac{\partial \Pi_M(w,\Delta)}{\partial \Delta}\right|_{\Delta=\Delta^2} - \left.\frac{\partial \Pi_M(w,\Delta)}{\partial \Delta}\right|_{\Delta=\Delta^1}\right] \le 0$$

…. (L20.1)

From above,

$$\frac{\partial^2 \Pi_M(w,\Delta)}{\partial w^2} < 0$$

Then if $w^2>w^1$,

$$\left.\frac{\partial \Pi_M(w,\Delta)}{\partial w}\right|_{w=w^2} < \left.\frac{\partial \Pi_M(w,\Delta)}{\partial w}\right|_{w=w^1}$$

and if $w^2<w^1$,

$$\left.\frac{\partial \Pi_M(w,\Delta)}{\partial w}\right|_{w=w^2} > \left.\frac{\partial \Pi_M(w,\Delta)}{\partial w}\right|_{w=w^1}$$

So, the first term in L20.1 is < 0 unless $w^2=w^1$, for which it is = 0. Similarly, the second term in L20.1 is < 0 unless $\Delta^2=\Delta^1$, for which it is = 0. Therefore, L20.1 is ≤ 0 for every (w^1,Δ^1) and (w^2,Δ^2) in the allowable region and this inequality is strict for every $(w^1,\Delta^1)\neq(w^2,\Delta^2)$. Using the above theorem, $\Pi_M(w,\Delta)$, restricted to $p_S+c_S<w+\Delta<w_{crit}$ and $\Delta>0$, is a strictly concave function of w and Δ.

(ii) From Lemma 19 (iv), for any $w<w_{crit}$, $0<\Delta^*(w)<\Delta_{crit}$, so $0<\Delta^*<\Delta_{crit}$ If $w=p_S+c_S$, then in effect there is no quantity premium and Δ is the constant wholesale price per unit. From 19(i), the manufacturer chooses a positive quantity premium. Therefore $w^*>p_S+c_S$. At w_{crit}, the manufacturer does not offer any quantity premium, i.e. $\Delta^*(w_{crit})=0$, as the manufacturer's capacity dictates the supply chain capacity for any positive quantity. We therefore only need to consider the case of $w^*=w_{crit}$ and $\Delta^*=0$. However, this is the same as a constant wholesale price schedule and from Lemma 11(ii) w* is strictly less than w_{crit}. So $0<w^*<w_{crit}$.

Lemma 21:

(i) If $F_X(x)$ satisfies the concavity condition given by equation (2), then the total expected channel profit when the manufacturer chooses both a wholesale price and a quantity premium is strictly greater than the total channel profit when the manufacturer only chooses a wholesale price but it is strictly less than the total supply chain profit when the supply chain is completely coordinated.

Proof:

Letting $\Pi_M(w)$ be the manufacturer's profit as a function of w when no quantity premium is allowed, then, using the expressions for $\partial\Pi_M(w,\Delta)/\partial w$ and $d\Pi_M(w)/dw$ given in Lemmas 20 and 10 respectively,

$$\frac{\partial\Pi_M(w,\Delta)}{\partial w} = +\Delta\left(\frac{dy_S(w)}{dw}\right)\left[1-F_X[y_S(w)]\right]+\left(\left.\frac{d\Pi_M(w)}{dw}\right|_{w'=w+\Delta}\right)$$

Let (w^*,Δ^*) be the optimal (w,Δ) pair chosen by the manufacturer and w^{**} be the optimal w chosen when no quantity premium is allowed. From Lemma 21 and Lemma 10, the following first order conditions must be satisfied,

(w^*,Δ^*): $\left.\frac{\partial\Pi_M(w,\Delta)}{\partial w}\right|_{w=w^*,\Delta=\Delta^*}=0$ $\left.\frac{\partial\Pi_M(w,\Delta)}{\partial \Delta}\right|_{w=w^*,\Delta=\Delta^*}=0$

w^{**}: $\left.\frac{d\Pi_M(w)}{dw}\right|_{w=w^{**}}=0$

But,

$$\left.\frac{\partial\Pi_M(w,\Delta)}{\partial w}\right|_{w=w^*,\Delta=\Delta^*} = \Delta^*\left(\left.\frac{dy_S(w)}{dw}\right|_{w=w^*}\right)\left[1-F_X[y_S(w^*)]\right]+\left(\left.\frac{d\Pi_M(w)}{dw}\right|_{w'=w^*+\Delta^*}\right)$$

Now,

(i) From Lemma 19(i) $\Delta^*>0$.

(ii) $\frac{dy_S(w)}{dw}>0$ from Lemma 4(ii)

(iii) $F_X(y_S(w^*)) < 1$

So,

$$\left.\frac{d\Pi_M(w)}{dw}\right|_{w'=w^*+\Delta^*}<0 \text{ and } \left.\frac{d\Pi_M(w)}{dw}\right|_{w'=w^{**}}=0$$

But, from Lemma 10, $\Pi_M(w)$ is a strictly concave function so $w^{**}<w^*+\Delta^*$. From Lemma 20, for any w, $\Delta^*(w)<\Delta_{crit}$. Therefore at w^*, $\Delta^*=\Delta^*(w^*)<\Delta_{crit}$, or $w^*+\Delta^*<w_{crit}$. Therefore, the optimal channel capacity is strictly greater when a quantity premium can be offered and thus the total channel profits, which depend only on the channel capacity, are strictly greater when a quantity premium can be offered.

As $w^*+\Delta^*<w_{crit}$, the total supply chain capacity is strictly less than that chosen by a central decision-maker.

Lemma 22:

The following continuous quantity premium price schedule is an optimal wholesale price schedule for the manufacturer,

$$\frac{dW(Q)}{dQ}=\frac{c_S-v_S F_X(Q)}{1-F_X(Q)}+p_S$$

Furthermore, it completely coordinates the supply chain but leaves the supplier with an expected profit of zero.

Proof:

For a wholesale price schedule specified by $W'(Q)$, let $\Pi_S(y,W'(Q))$ be the supplier's expected profit from choosing a capacity of y if the manufacturer has infinite capacity.

$$\Pi_S(y,W'(Q))=-c_S y+\int_a^y \left(W(x)-p_S x\right)f_X(x)dx+\left(W(y)-p_S y\right)[1-F_X(y)]$$
$$+v_S\int_a^y [y-x]f_X(x)dx$$

So,

$$\frac{d\Pi_S(y,W'(Q))}{dy}=-\left(c_S-v_S F_X(y)\right)+\left(W'(y)-p_S\right)\left[1-F_X(y)\right]$$

Suppose,

$$W'(Q)=\frac{c_S-v_S F_X(Q)}{1-F_X(Q)}+p_S$$

Then,

$$\frac{d\Pi_S(y,W'(Q))}{dy}=-\left(c_S-v_S F_X(y)\right)+\left(c_S-v_S F_X(y)\right)=0\quad\forall y$$

So the supplier receives an expected profit of zero for all capacity choices.

For a wholesale price schedule specified by $W'(Q)$, let $\Pi_M(y,W'(Q))$ be the manufacturer's expected profit from choosing a capacity of y if the supplier has infinite capacity.

$$\Pi_M(y, W'(Q)) = -c_M y + \int_a^y \left((r - p_M)x - W(x)\right) f_X(x)dx + \left((r - p_M)y - W(y)\right)[1 - F_X(y)]$$
$$+ v_M \int_a^y [y - x] f_X(x)dx$$

So,

$$\frac{d\Pi_M(y, W'(Q))}{dy} = \left(r - p_M - W'(y)\right)[1 - F_X(y)] - [c_M - v_M F_X(y)]$$

and,

$$\frac{d^2\Pi_M(y, W'(Q))}{dy^2} = -\left(r - p_M - v_M - W'(y)\right) f_X(y) - W''(y)[1 - F_X(y)]$$

Suppose,

$$W'(Q) = \frac{c_S - v_S F_X(Q)}{1 - F_X(Q)} + p_S$$

Then,

$$W''(Q) = \frac{(c_S - v_S) f_X(Q)}{(1 - F_X(Q))^2}$$

Therefore,

$$\begin{aligned}
\frac{d\Pi_M(y, W'(Q))}{dy} &= \left(r - p_M - p_S - \frac{c_S - v_S F_X(y)}{1 - F_X(y)}\right)[1 - F_X(y)] - [c_M - v_M F_X(y)] \\
&= [1 - F_X(y)](r - p_M - p_S) - c_S + v_S F_X(y) - c_M + v_M F_X(y) \\
&= (r - p_M - p_S - c_M - c_S) - F_X(y)(r - p_M - p_S - v_M - v_S)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2\Pi_M(y, W'(Q))}{dy^2} &= -\left(r - p_M - p_S - v_M - \frac{c_S - v_S F_X(y)}{1 - F_X(y)}\right) f_X(y) \\
&\quad - \frac{(c_S - v_S) f_X(y)}{(1 - F_X(y))^2}[1 - F_X(y)] \\
&= -\left(r - p_M - p_S - v_M - \frac{c_S - v_S F_X(y)}{1 - F_X(y)}\right) f_X(y) \\
&\quad - \frac{(c_S - v_S) f_X(y)}{1 - F_X(y)} \\
&= -\frac{f_X(y)}{1 - F_X(y)}\left([1 - F_X(y)](r - p_M - p_S - v_M - v_S)\right) \\
&= -f_X(y)(r - p_M - p_S - v_M - v_S) < 0
\end{aligned}$$

as r>p_M+p_S+v_S+v_S.

So, $\Pi_M(y,W'(Q))$ is concave in y and the first order condition is sufficient for optimality. The first order condition is given by,

$$\frac{d\Pi_M(y,W'(Q))}{dy}=0 \Leftrightarrow F_X(y)=\left(\frac{r-p_M-p_S-c_M-c_S}{r-p_M-p_S-v_M-v_S}\right) \Leftrightarrow y^*=y_I$$

The manufacturer's optimal choice is the same as that of a central-decision-maker. The supplier is indifferent to its capacity choice from above, so it is willing to invest in this capacity also. The supply chain is completely coordinated. The manufacturer captures the total expected supply chain profit as the supplier's expected profit is zero.

Note that one can induce the supplier to choose y_I by introducing an arbitrarily small quadratic penalty into the wholesale price schedule.

Lemma 23:

In the N supplier exogenous wholesale price game, the manufacturer and suppliers choose their capacities to be $y_M^*=y_{S1}^*=\ldots=y_{SN}^*=min\{y_S^{min}(w_1,\ldots,w_N),y_M(w_{Tot})\}$.

Proof:

Supplier N never chooses a capacity larger than the supplier N-1's capacity choice, as its total sales are limited by the this capacity. From Lemma 3, supplier N's profit, $\Pi_{SN}(y_N)$, is concave with the maximum achieved at $y_{SN}(w_N)$. For $y_N<y_{SN}(w_N)$, $\Pi_{SN}(y_N)$ is increasing in y. Therefore, if supplier N-1 announces a capacity of $y_{N\text{-}1}$, then supplier N chooses a capacity of $min\{y_{N\text{-}1},y_{SN}(w_N)\}$. Supplier N-1 therefore does not choose a capacity larger than $y_{SN}(w_N)$. Repeating this argument for suppliers N-1 through 1, the suppliers choose their capacities equal to $min\{y_M,y_{S1}(w_1),\ldots,y_{SN}(w_N)\}$, where y_M is the capacity announced by the manufacturer. From Lemma 2, the manufacturer's profit, $\Pi_M(y)$, is concave with the maximum achieved at $y_M(w_{Tot})$. For $y<y_M(w_{Tot})$, $\Pi_M(y)$ is increasing in y. The manufacturer does not choose a capacity larger than the suppliers would be willing to choose. Therefore, the manufacturer chooses its capacity, $y_M^*=min\{y_M(w_{Tot}),y_{S1}(w_1),\ldots,y_{SN}(w_N)\}$. The suppliers then choose their capacities such that $y_{S1}^*=\ldots=y_{SN}^*=min\{y_M(w_{Tot}),y_{S1}(w_1),\ldots,y_{SN}(w_N)\}$. By definition $y_S^{min}(w_1,\ldots,w_N)=min_n\{y_{Sn}(w_n)$.

Lemma 24:

(i) $y_M(w_{Tot})=y_I*$, the coordinated channel capacity, iff $w_{Tot} = w_{Tot}^{crit}$, where

$$w_{Tot}^{crit} = \frac{(r-p_M)(c_s^{Tot}-v_s^{Tot})+(c_M-v_M)p_s^{Tot}+c_M v_s^{Tot}-v_M c_s^{Tot}}{c_M-v_M+c_s^{Tot}-v_s^{Tot}}$$

and $y_M(w_{Tot})\geq y_I*$ iff $w_{Tot} \leq w_{Tot}^{crit}$

(ii) $y_{Sn}(w_n)=y_I*$ iff $w_n = w_n^{crit}$,where

$$w_n^{crit} = \frac{(r-p_M-p_s^{Tot})(c_{S_n}-v_{S_n})+p_{S_n}(c_M-v_M+c_s^{Tot}-v_s^{Tot})-c_S(v_M+v_s^{Tot})+v_S(c_M+c_s^{Tot})}{c_M-v_M+c_s^{Tot}-v_s^{Tot}}$$

and $y_{Sn}(w_n)\geq y_I*$ iff $w_n \geq w_n^{crit}$.

(iii) $\sum_{n=1}^{N} w_n^{crit} = w_{Tot}^{crit}$

(iv) Complete channel coordination occurs iff $w_n=w_n^{crit}$ $n=1,\ldots,N$.

Proof:

(i) $y_M(w_{Tot}) = F_X^{-1}\left(\frac{m_M-c_M}{m_M-v_M}\right) \quad y_{\mathrm{I}}^* = F_X^{-1}\left(\frac{m_I-c_I}{m_I-v_I}\right)$ where $m_M=r-w_{Tot}-p_M$, $m_I=r-p_M-p_S^{Tot}$, $c_I=c_M+p_S^{Tot}$, and $v_I=v_M+v_S^{Tot}$. So, $y_M(w)\geq y_I*$ iff

$$\left(\frac{m_M-c_M}{m_M-v_M}\right)\geq\left(\frac{m_I-c_I}{m_I-v_I}\right)$$

or,

$$w_{Tot} \leq \frac{(r-p_M)(c_s^{Tot}-v_s^{Tot})+(c_M-v_M)p_s^{Tot}+c_M v_s^{Tot}-v_M c_s^{Tot}}{c_M-v_M+c_s^{Tot}-v_s^{Tot}}$$

(ii)

$y_{S_n}(w_n) = F_X^{-1}\left(\frac{m_{S_n}-c_{S_n}}{m_{S_n}-v_{S_n}}\right) \quad y_{\mathrm{I}}^* = F_X^{-1}\left(\frac{m_I-c_I}{m_I-v_I}\right)$ where $m_{Sn}=w_n-p_n$. So, $y_S(w)\geq y_I*$ iff

$$\left(\frac{m_{S_n}-c_{S_n}}{m_{S_n}-v_{S_n}}\right)\geq\left(\frac{m_I-c_I}{m_I-v_I}\right)$$

or,

$$w_n \geq \frac{(r-p_M-p_s^{Tot})(c_{S_n}-v_{S_n})+p_{S_n}(c_M-v_M+c_s^{Tot}-v_s^{Tot})-c_{S_n}(v_M+v_s^{Tot})+v_{S_n}(c_M+c_s^{Tot})}{c_M-v_M+c_s^{Tot}-v_s^{Tot}}$$

(iii)

$$\sum_{n=1}^{N} w_n^{crit}$$

$$= \sum_{n=1}^{N}\left(\frac{(r-p_M-p_s^{Tot})(c_{S_n}-v_{S_n})+p_{S_n}(c_M-v_M+c_s^{Tot}-v_s^{Tot})-c_{S_n}(v_M+v_s^{Tot})+v_{S_n}(c_M+c_s^{Tot})}{c_M-v_M+c_s^{Tot}-v_s^{Tot}}\right)$$

$$=\frac{(r-p_M)(c_S^{Tot}-v_S^{Tot})+(c_M-v_M)p_S^{Tot}+c_M v_S^{Tot}-v_M c_S^{Tot}}{c_M-v_M+c_s^{Tot}-v_s^{Tot}}=w_{Tot}^{crit}$$

(iv) Complete channel coordination occurs iff the supply chain capacity choice is equal to y_I*. In other words, from Lemma 37, one needs $\min\{y_{S1}(w_1),..,y_{SN}(w_N),y_M(w_{Tot})\}=y_I*$. This is true iff (a) $y_M(w_{Tot})\geq y_I*$, (b) $y_{Sn}(w_n)\geq y_I*$, n=1,…,N and (c) one of the inequalities in (a) or (b) to hold with equality. From (i) above, $y_M(w_{Tot})\geq y_I*$ iff $w_{Tot}\leq w_{Tot}^{crit}$. So (a) holds iff $w_{Tot}\leq w_{Tot}^{crit}$. From (ii) above, $y_{Sn}(w_n)\geq y_I*$, n=1,…,N iff $w_n\geq w_n^{crit}$ n=1,…,N. So (b) holds iff $w_n\geq w_n^{crit}$ n=1,…,N. Therefore, using (iii) above, (a) and (b) both hold iff $w_n\geq w_n^{crit}$ n=1,…,N and $w_{Tot}=w_{Tot}^{crit}$. However, if some $w_n>w_n^{crit}$, then $w_{Tot}=w_{Tot}^{crit}$ iff some other $w_n<w_n^{crit}$. This cannot be if (a) and (b) both hold. Therefore (a) and (b) both hold iff $w_n=w_n^{crit}$ n=1,…,N, which implies $w_{Tot}=w_{Tot}^{crit}$. For these wholesale prices the inequalities in (a) and (b) all hold with equality so (c) holds.

Lemma 25:

In the N supplier wholesale price game, if $F_X(x)$ satisfies the concavity condition, then
(i) The manufacturer's expected profit as a function of the suppler 1 wholesale price w, $\Pi^N_M(w)$, restricted to $p_{S1}+c_{S1}<w\leq w_1^{crit}$, is a strictly concave function.
(ii) The optimal wholesale prices w_n* are strictly less than w_n^{crit}, n=1,…,N.
(iii) The channel fails to be completely coordinated.

Proof

(i)

$$\Pi_M^N(w) = -c_M y_{S_1}(w) + (r - p_M - G_1(w)) \int_a^{y_{S_1}(w)} x f_X(x)dx$$

$$+ (r - p_M - G_1(w)) y_S(w)[1 - F_X(y_{S_1}(w))] + v_M \int_a^{y_{S_1}(w)} [y_{S_1}(w) - x] f_X(x)dx$$

Taking the derivative with respect to *w*,

$$\frac{d\Pi_M^N(w)}{dw} = \left(\frac{dy_{S_1}(w)}{dw}\right)\Big((r - p_M - G(w) - c_M) - (r - p_M - G(w) - v_M)F_X(y_{S_1}(w))\Big)$$

$$- \mathrm{A}_1 \int_{\mathrm{a}}^{y_{S_1}(w)} x f_X(x)dx - \mathrm{A}_1 y_{S_1}(w)[1 - F_X(y_{S_1}(w))]$$

Taking the second derivative,

$$\frac{d^2\Pi_M^N(w)}{dw^2} = \left(\frac{d^2 y_{S_1}(w)}{dw^2}\right)\Big((r - p_M - G(w) - c_M) - (r - p_M - G(w) - v_M)F_X(y_{S_1}(w))]\Big)$$

$$- 2A_1\left(\frac{dy_{S_1}(w)}{dw}\right)[1\text{-} F_X(y_{S_1}(w)] - (r - p_M - G(w) - v_M)\ f_X(y_{S_1}(w))\left(\frac{dy_{S_1}(w)}{dw}\right)^2$$

Now,

(i) From Lemma 4(ii), $\frac{dy_{S_1}(w)}{dw} > 0$

(ii) As $F_X(x)$ satisfies the concavity condition, then $\frac{d^2 y_{S_1}(w)}{dw^2} \leq 0$

(iii) $r>w+p_M+c_M$ in the allowable range for w,

(iv) $c_M>v_M$

(v) $F_X(y_S(w))\leq(r\text{-}p_M\text{-}G(w)\text{-}c_M)/(r\text{-}p_M\text{-}G(w)\text{-}v_M)<1$

(vi) $A_1>0$

Using (i)-(vi), $\frac{d^2\Pi_M^N(w)}{dw^2} < 0$

(ii)

$$\left.\frac{d\Pi_M^N(w)}{dw}\right|_{w=w_1^{crit}} = -\mathrm{A}_1 \int_{\mathrm{a}}^{y_{S_1}(w_1^{crit})} x f_X(x)dx - \mathrm{A}_1 y_{S_1}(w_1^{crit})[1 - F_X(y_{S_1}(w_1^{crit}))] < 0$$

As $\Pi^N_M(w)$ is concave function, w^* must be strictly less than w_1^{crit}. The optimal wholesale price for supplier 1 is $w_1^*=w^*$. The optimal wholesale prices for $w_2,\ldots,w_N$, are given by $T^n_1(w_n^*)=w_1^*$, $n=2,\ldots,N$. $T^n_1(w_n)$ is a one-to-one mapping such that $y_{Sn}(w_n)=y_{S1}(T^n_1(w_n))$. $T^n_1(w_\mathrm{n})$ is strictly increasing in w^n. $y_{Sn}(w_n^{crit})=y_{S1}(w_1^{crit})$, therefore $T^n_1(w_n^{crit})=w_1^{crit}$. As $w_1^*<w_1^{\mathrm{crit}}$, $w_\mathrm{n}^*<w_1^{crit}$, $n=1,\ldots,N$.

(iii) From Lemma 38(iv), the channel is only coordinated if $w_n=w_n^{crit}$ $n=1,\ldots,N$. Therefore, the channel is not completely coordinated when the manufacturer sets the prices to maximize its expected profit

Lemma 26:

(i) If $\beta_S<(r-c)/(r-v)$, then a central decision maker would invest in either medium or large capacity, investing in medium capacity if $\beta_L\leq(c-v)/(r-v)$ and in large capacity otherwise.

(ii) If $\beta_S<(r-c)/(r-v)$ and $\beta_\mathrm{L}\leq(c-v)/(r-v)$, then a quantity premium price schedule of (c,Δ_M^*) induces the supplier to invest in medium capacity, where $\Delta^*_M = \left(\frac{\beta_S}{1-\beta_S}\right)(c-v)$. Furthermore, the channel is completely coordinated and the manufacturer captures all the expected supply chain profit.

(iii) If $\beta_\mathrm{S}<(r-c)/(r-v)$ and $\beta_\mathrm{L}>(c-v)/(r-v)$, then a quantity premium price schedule of $(c,\Delta_M^*,\Delta_L^*)$ induces the supplier to invest in large capacity, where Δ_M^* is given above and $\Delta^*_L = \left(\frac{1-\beta_L}{\beta_L} - \frac{\beta_S}{1-\beta_S}\right)(c-v)$. Furthermore, the channel is completely coordinated and the manufacturer captures all the expected supply chain profit.

Proof:

(i) Let $\Pi_I(K)$ denote the expected profit obtained by a central decision-maker that invests in a capacity of K. Then,

$$\Pi_I(S) = -cS + rS$$
$$\Pi_I(M) = -cM + \beta_S\left[rS + v(M-S)\right] + (1-\beta_S)rM$$
$$\Pi_I(L) = -cL + \beta_S\left[rS + v(L-S)\right] + \beta_M\left[rM + v(L-M)\right] + \beta_L L$$

A central decision-maker does not invest in small capacity if $\Pi_I(M)>\Pi_I(S)$.

$$\Pi_I(M) > \Pi_I(S) \Leftrightarrow -cM + \beta_S\left[rS + v(M-S)\right] + (1-\beta_S)rM > -cS + rS$$
$$\Leftrightarrow \beta_S < \frac{r-c}{r-v}$$

If $\beta_S<(r-c)/(r-v)$, then a central decision maker invests in either medium or large capacity. For this range of β_S it invests in medium capacity iff $\Pi_I(M)>\Pi_I(L)$.

$$\Pi_I(M) > \Pi_I(L) \Leftrightarrow -c(L-M) + (\beta_S + \beta_M)v(L-M) + \beta_L r(L-M) < 0$$
$$\Leftrightarrow \beta_L < \frac{c-v}{r-v}$$

(ii) Let the manufacturer offer the following quantity premium price schedule, (c,Δ_M,Δ_L). Let $\Pi_S(K,c,\Delta_M,\Delta_L)$ denote the expected profit obtained by the supplier if it invests in a capacity of K, for this price schedule.

$$\begin{aligned}
\Pi_S(S,c,\Delta_M,\Delta_L) &= -cS + cS = 0\\
\Pi_S(M,c,\Delta_M,\Delta_L) &= -cM + \beta_S\left[cS + v(M-S)\right] + (1-\beta_S)\left[cS + (c+\Delta_M)(M-S)\right]\\
\Pi_S(L,c,\Delta_M,\Delta_L) &= -cL + \beta_S\left[cS + v(L-S)\right] + \beta_M\left[cS + (c+\Delta_M)(M-S) + v(L-M)\right]\\
&\quad + \beta_L\left[cS + (c+\Delta_M)(M-S) + (c+\Delta_M+\Delta_L)(L-M)\right]\\
&= (r-v)M - (c-v)B - \beta_S(r-v)(M-S) + \beta_L(r-v)(B-M)
\end{aligned}$$

The supplier prefers (or be indifferent to) M to S, if $\Pi_S(M,c,\Delta_M,\Delta_L)\geq\Pi_S(S,c,\Delta_M,\Delta_L)$.

$$\Pi_S(M,c,\Delta_M,\Delta_L) \geq \Pi_S(S,c,\Delta_M,\Delta_L) \Leftrightarrow \Delta_M \geq \left(\frac{\beta_S}{1-\beta_S}\right)(c-v) \geq 0$$

The supplier prefers (or be indifferent to) L to M, if $\Pi_S(L,c,\Delta_M,\Delta_L)\geq\Pi_S(M,c,\Delta_M,\Delta_L)$.

$$\Pi_S(L,c,\Delta_M,\Delta_L) \geq \Pi_S(M,c,\Delta_M,\Delta_L) \Leftrightarrow \Delta_M + \Delta_L \geq \left(\frac{1-\beta_L}{\beta_L}\right)(c-v)$$
$$\Leftrightarrow \Delta_L \geq \left(\frac{1-\beta_L}{\beta_L} - \frac{\beta_S}{1-\beta_S}\right)(c-v) \geq 0$$

If $\beta_S<(r-c)/(r-v)$, $\beta_B\leq(c-v)/(r-v)$ and the manufacturer offers a quantity premium price schedule of $(c,\Delta_M{*},0)$, then $\Pi_S(S,c,\Delta_M,0)=\Pi_S(M,c,\Delta_M,0)>\Pi_S(L,c,\Delta_M,0)$, and the supplier invests in medium capacity. A central decision-maker would also invest in medium capacity and thus the channel is completely coordinated. The manufacturer's expected profit is given by,

$$\begin{aligned}
\Pi_M(M,c,\Delta_M,\Delta_L) &= \beta_S(r-c)S + (1-\beta_S)\left[(r-c)S + (r-c-\Delta_M^*)(M-S)\right]\\
&= -cM + \beta_S\left[rS + v(M-S)\right] + (1-\beta_S)rM\\
&= \Pi_I(M)
\end{aligned}$$

and thus the manufacturer's expected profit is the same as the total expected supply chain profit obtained by a central decision-maker. The manufacturer can do no better than this and is thus this quantity premium price schedule of (c,Δ_M*,0) [or (c,Δ_M*)] is optimal for the manufacturer.

(iii) If β_S<(r-c)/(r-v), $\beta_L \le (c-v)/(r-v)$ and the manufacturer offers a quantity premium price schedule of (c,Δ_M*,Δ_L*), then $\Pi_S(S,c,\Delta_M^*,\Delta_L^*)=\Pi_S(c,\Delta_M^*,\Delta_L^*)=\Pi_S(c,\Delta_M^*,\Delta_L^*)$, and the supplier invests in large capacity. A central decision-maker would also invest in large capacity and thus the channel is completely coordinated. The manufacturer's expected profit is given by,

$$\begin{aligned}\Pi_M(L,c,\Delta_M,\Delta_L) &= \beta_S(r-c)S + \beta_M\left[(r-c)S + (r-c-\Delta_M^*)(M-S)\right] \\ &\quad + \beta_L\left[(r-c)S + (r-c-\Delta_M^*)(M-S) + (r-c-\Delta_M^* - \Delta_L^*)(L-M)\right] \\ &= (r-v)M - (c-v)B - \beta_S(r-v)(M-S) - (1-\beta_L)(r-v)(L-M) \\ &= \Pi_M(L)\end{aligned}$$

and thus the manufacturer's expected profit is the same as the total expected supply chain profit obtained by a central decision-maker. The manufacturer can do no better than this and is thus this quantity premium price schedule of (c,Δ_M*,Δ_L*) is optimal for the manufacturer.

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5.2 Appendix 2

This appendix contains proofs of the lemmas from Chapter 3.

Lemma 1:

(i) A lower bound for the minimum shortfall in problem **P1** is given by problem **P2**,

$$\underset{M}{\text{Max}}\ \{\sum_{i\in M} d_i - \min_{L_1,..L_K}\{\sum_{k=1}^{K}\sum_{j\in P^k(L^k)} c_j^k\}\}$$

subject to

(i) $M \subseteq \{\varnothing,1,\dots,I\}$

(ii) $L_k \cap L_{k'} = \varnothing \quad \forall k \neq k'$

(iii) $\bigcup_{k=1}^{K} L_k = M$

(ii) If either the number of stages K or the number of products I is less than three, then the minimum shortfall in problem **P1** is equal to the lower bound in (i).

Proof:

(i) **P1** is given by,

$$\min_{\mathbf{x},\mathbf{s}}\{\sum_{i=1}^{I} s_i\}$$

subject to

1. $\displaystyle\sum_{j\in P^k(i)} x_{ij}^k + s_i \geq d_i \qquad i=1,\dots,I \quad k=1,\dots,K$
2. $\displaystyle\sum_{i\in Q^k(j)} x_{ij}^k \leq c_j^k \qquad j=1,\dots,J_k \quad k=1,\dots,K$

$\mathbf{x},\mathbf{s} \geq \mathbf{0}$

Let π_i^k be the dual variables for the Type 1 constraints and μ_j^k be the dual variables for the Type 2 constraints. Letting $\nu_j^k = -\mu_j^k$ gives us the following dual problem **D1**,

$$\underset{\boldsymbol{\pi},\boldsymbol{\nu}}{\text{Max}}\ \{\sum_{k=1}^{K}\sum_{i=1}^{I}\pi_i^k d_i - \sum_{k=1}^{K}\sum_{j=1}^{J_k}\nu_j^k c_j^k\}$$

subject to

1. $\displaystyle\sum_{k=1}^{K}\pi_i^k \leq 1 \qquad i=1,\dots,I$
2. $\pi_i^k \leq \nu_j^k \qquad j\in P^k(i) \quad i=1,\dots,I \quad k=1,\dots,K$
3. $\boldsymbol{\pi} \geq \mathbf{0} \quad , \quad \boldsymbol{\nu} \geq \mathbf{0}$

Let ***C*** be the set of solutions to **D1** that meets the following two conditions,

(i) $\boldsymbol{\pi}, \boldsymbol{\upsilon} \in \{0,1\}$

(ii) $\nu_j^k = \begin{cases} 1 \text{ if } \pi_i^k = 1 \text{ for some } i \in Q^k(j) \\ 0 \text{ otherwise} \end{cases}$

Each element in ***C*** is a feasible solution to **D1**.

Let ***E*** be the set of feasible solutions to **P2**. A feasible solution ($M,L_1,..,L_K$) to **P2** has the following objective value,

$$\sum_{i \in M} d_i - \sum_{k=1}^{K} \sum_{j \in P^k(L^k)} c_j^k$$

There is a one-to-one correspondence between elements in ***C*** and elements in ***E***; each element in ***C*** has a corresponding element in ***E*** with the same objective value and vice versa. To see this, consider an element ($M,L_1,..,L_K$) of ***E***. This can be mapped into an element of ***C*** as follows. For k=1,…,K set $\pi_i^k = 1 \ \forall\, i \in L_k$, $\pi_i^k = 0 \ \forall\, i \notin L_k$, $\nu_j^k = 1 \ \forall\, j \in P^k(L_k)$ and $\nu_j^k = 0 \ \forall\, j \notin P^k(L_k)$.

The objective value this element of ***C*** is,

$$\sum_{k=1}^{K} \sum_{i \in L^k} d_i - \sum_{k=1}^{K} \sum_{j \in P^k(L^k)} c_j^k = \sum_{i \in M} d_i - \sum_{k=1}^{K} \sum_{j \in P^k(L^k)} c_j^k$$

This is the same as the objective value for ($M,L_1,..,L_K$). Similarly any element of ***C*** can be mapped into an element of ***E*** by setting L_k to be the set of all products i with $\pi_i^k = 1$ k=1,…,K. Note that because of the Type 1 constraints, at most one π_i^k can equal 1 for each i=1,...,I, so that $L_k \cap L_{k'} = \varnothing \ \forall k \neq k'$. Again the objective value of this element of ***E*** is equal to the objective value of the element of ***C***.

Therefore, each feasible solution to **P2** corresponds to a feasible solution to **D1.** The objective value of such a solution gives a lower bound on the optimal value of **D1,** and hence from duality a lower bound on the minimum shortfall objective value of **P1**. The optimum value to **P2** is the maximum such lower bound.

(ii) From duality, the optimal solution to **P1** must equal the optimal solution to **D1**. From part (i), **P2** gives the optimal solution to **D1** subject to,

(i) $\boldsymbol{\pi}, \boldsymbol{\upsilon} \in \{0,1\}$

(ii) $\nu_j^k = \begin{cases} 1 \text{ if } \pi_i^k = 1 \text{ for some } i \in Q^k(j) \\ 0 \text{ otherwise} \end{cases}$

If the optimal solution to **D1** can be shown to satisfy both (i) and (ii), then **P2** gives the optimal solution to **D1** and hence the optimal solution to **P1**.

Let (π,ν) be a feasible solution to **D1** such that some $\nu_j^k > \pi_i^k \ \forall\ i \in Q^k(j)$. The objective function can increased by decreasing ν_j^k until $\nu_j^k = \max_{i \in Q^k(j)} \{\pi_i^k\}$ without violating any constraint. Decreasing ν_j^k any further violates a Type 2 constraint. Therefore, an optimal solution must satisfy $\nu_j^k{}^* = \max_{i \in Q^k(j)} \{\pi_i^k{}^*\}$. Assume for the moment that the optimal solution to **D1** is binary, i.e. satisfies condition (i). Then, condition (ii) must hold as $\nu_j^k{}^* = \max_{i \in Q^k(j)} \{\pi_i^k{}^*\}$.

All that remains to be shown is that the optimal solution to **D1** is binary. Substitute $y_j^k = 1 - \nu_j^k$ into problem **D1**. The following problem **D2** is obtained.

$$\operatorname*{Max}_{\boldsymbol{\pi},\boldsymbol{\nu}} \ \{\sum_{k=1}^{K}\sum_{i=1}^{I}\pi_i^k d_i + \sum_{k=1}^{K}\sum_{j=1}^{J_k} y_j^k c_j^k - \sum_{k=1}^{K}\sum_{j=1}^{J_k} c_j^k\}$$

subject to

1. $\sum_{k=1}^{K}\pi_i^k \le 1 \qquad i = 1,\ldots,I$
2. $\pi_i^k + y_j^k \le 1 \qquad j \in P^k(i) \quad i = 1,\ldots,I \quad k = 1,\ldots,K$
3. $y_j^k \le 1 \qquad j = 1,\ldots,J_k \quad k = 1,\ldots,K$
4. $\boldsymbol{\pi} \ge \mathbf{0}$

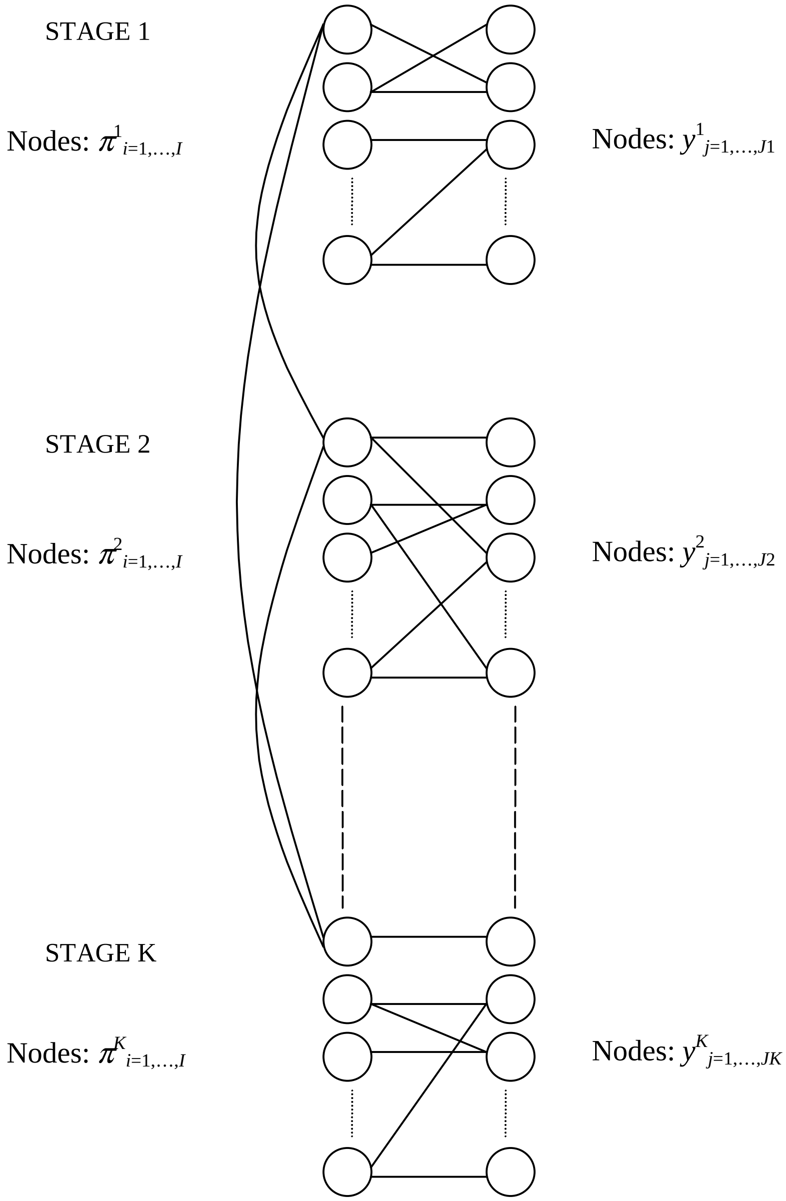
If y_j^k is negative, then the Type 2 constraints in which it appears are satisfied with strict inequality because from the Type 1 constraints $\pi_i^k \le 1$. Set y_j^k to zero. This solution remains feasible. The objective function is strictly increased. Therefore **D2** can be restricted to $\mathbf{y} \ge \mathbf{0}$ without any affect on the optimal solution. The upper bound constraints on the **y** variables (Type 3) can be ignored as the non-negativity of the $\boldsymbol{\pi}$ variables along with the Type 2 constraints ensure that the **y** upper bounds are not exceeded. Therefore **D2** can be solved by the following problem **D3**,

$$\operatorname*{Max}_{\boldsymbol{\pi},\boldsymbol{\nu}} \ \{\sum_{k=1}^{K}\sum_{i=1}^{I}\pi_i^k d_i + \sum_{k=1}^{K}\sum_{j=1}^{J_k} y_j^k c_j^k - \sum_{k=1}^{K}\sum_{j=1}^{J_k} c_j^k\}$$

subject to

1. $\sum_{k=1}^{K}\pi_i^k \le 1 \qquad i = 1,\ldots,I$
2. $\pi_i^k + y_j^k \le 1 \qquad j \in P^k(i) \quad i = 1,\ldots,I \quad k = 1,\ldots,K$
3. $\boldsymbol{\pi} \ge \mathbf{0} \ , \ \mathbf{y} \ge \mathbf{0}$

Let **A** be the constraint matrix of the linear program **D3**. **A** is the clique matrix of the undirected graph **G**, in which the π and **y** variables are nodes. Figure 1 shows an example of the graph **G**.



Note: The $(\pi^k_i, \pi^{k'}_i)$ arcs are only shown for i=1

Figure 1
The Constraint Matrix A is the Clique Matrix of the above Graph, G

A π_i^k variable is a node for product i at stage k and a y_j^k variable is a node for the j^{th} plant of stage k. An arc joins π_i^k to y_j^k $\forall j \in P^k(i)$, i.e. the product node i at stage k to all plants that can process product i at stage k. For each product i, there is an arc $(\pi_i^k, \pi_i^{k'})$ from node π_i^k to $\pi_i^{k'}$ $k' > k$, k=1,…,K. In other words the $\boldsymbol{\pi}$ variables for product i have arcs to all other $\boldsymbol{\pi}$ variables for product i. Note that I use the convention that $(\pi_i^k, \pi_i^{k'})$ arcs always have the smaller k value as the first node of the arc. There are no arcs joining the $\boldsymbol{\pi}$ variables of two different products. The π_i^k k=1,…,K for each product i form a clique as does each π_i^k to y_j^k arc.

If the number of stages K is less than three, then **G** is a bipartite graph. To see this, group all π_i^1 and y_j^2 nodes in Set 1. Group all π_i^2 and y_j^1 nodes in Set 2. The only arcs in **G** are those joining a node in Set 1 to a node in Set 2. As **G** is bipartite, it is a perfect graph (Nemhauser and Wolsley, 1988).

If the number of products I is less than three, then the graph G is again perfect. The proof of for this case is a little more involved and is most easily understood by referring to Figure 1.

- Consider a cycle with no $(\pi_1^k, \pi_1^{k'})$ type arcs and no $(\pi_2^k, \pi_2^{k'})$ type arcs. This cycle must contain only (π_i^k, y_j^k) type arcs for a single stage k. Such a cycle must have an even number of arcs as each stage k's subgraph is a bipartite.
- There is no cycle with exactly one $(\pi_1^k, \pi_1^{k'})$ type arc and no $(\pi_2^k, \pi_2^{k'})$ type arcs as such a cycle would leave the set of $\boldsymbol{\pi}$ nodes for stage k and never return. Likewise there is no cycle with exactly one $(\pi_2^k, \pi_2^{k'})$ type arc and no $(\pi_1^k, \pi_1^{k'})$ type arcs.
- Consider a cycle exactly one $(\pi_1^{k_1}, \pi_1^{k_1'})$ type arc and exactly one $(\pi_2^{k_2}, \pi_2^{k_2'})$ type arc, where the subscript on the stage k denotes the product i to which it refers. We must have k_1=k_2 and k_1'=k_2' as otherwise the "cycle" would leave the set of $\boldsymbol{\pi}$ nodes for stage k_1 and never return. Clearly this would not be a cycle. Any cycle in which k_1=k_2 and k_1'=k_2' must have an even number of arcs.

Therefore, any cycle that contains no more than one $(\pi_1^k, \pi_1^{k'})$ type arc and no more than one $(\pi_2^k, \pi_2^{k'})$ type arcs must have an even number of arcs. So any odd length cycle must contain at least two $(\pi_i^k, \pi_i^{k'})$ type arcs for i=1 or 2. Consider a cycle containing at least two $(\pi_i^k, \pi_i^{k'})$ type arcs. The only clique of graph **G** that contains $(\pi_i^k, \pi_i^{k'})$ arcs is the clique $\{\pi_i^1, \pi_i^2, \ldots, \pi_i^K\}$. This clique contains all such $(\pi_i^k, \pi_i^{k'})$ arcs and therefore contains another arc of the cycle. Therefore from Theorem 5.17 of Nemhauser and Wolsley (1988), the graph **G** is perfect.

So if either the number of products K or number of stages I is less than three, then **G** is a perfect graph. Therefore the polyhedron defined by **D3** (Ax≤1) is integral. The optimal solution to **D3** is thus integral. In fact it is binary, because of the right hand side values. This in turn implies that the optimal solution to **D1** is binary.

Lemma 2:

(i) For any subset of products, M, define the problem **P3(*M*)** as

$$\underset{L_1,\ldots,L_k}{\text{Min}}\ \{\sum_{k=1}^{K}\sum_{j\in P^k(L^k)} c_j^k\}$$

subject to

(i) $L_k \cap L_{k'} = \emptyset \quad \forall k \neq k'$

(ii) $\bigcup_{k=1}^{K} L_k = M$

If for every possible M, there exists an optimal solution to **P3**(M) with only one non-empty L_k*, then a stage-spanning bottleneck can never occur in this case.

(ii) If $A_{\min} \geq \frac{TC_{\max}}{2}$, where $A_{\min} = \underset{\substack{i=1,\ldots,I \\ k=1,\ldots,K}}{\text{Min}}\ \{\sum_{j\in P^k(i)} c_j^k\}$ and $TC_{\max} = \underset{k=1,\ldots,K}{\text{Max}}\ \{TC_k\}$,then a stage-spanning bottleneck can never occur. Note that $A_{\min}$ is the minimum total capacity available to any product at any stage and $TC_{\max}$ is the maximum total stage capacity across all stages.

Proof:

(i) Let **d**=$\{d_1,\ldots,d_I\}$ be any demand realization. Let M* be the optimal M set for problem **P2** given this demand realization **d**. If for every possible subset of products, M, there exists an optimal solution to **P3**(M) with only one non-empty L_k*, then for the optimal set M* there exists an optimal solution to **P3**(M*) with only one non-empty L_k*. **P3**(M) is the internal minimization in **P2** and therefore if **P3**(M*) has only one non-empty L_k*, then **P2** has only one non-empty L_k*. By definition a stage-spanning bottleneck does not occur for this demand realization, **d**. This is true for any demand realization and so a stage-spanning bottleneck can never occur.

(ii) For any subset of products, M, let $\Lambda(M)$ be the set of stages with non-empty L_k*, where L_k* are the minimizing sets in **P3**(M), subject to there being at least two non-empty L_k*. Then $|\Lambda(M)|\geq 2$.

$$\sum_{k=1}^{K}\sum_{j\in P^{k'}(L_k^*)} c_j^k \geq \sum_{k\notin\Lambda(M)} 0 + \sum_{k\in\Lambda(M)} A_{\min} \geq 2A_{\min}$$

Construct a set of L_k, k=1,..K, as follows. Set $L_1^{new} = \bigcup_{k=1}^{K} L_k^*$ and $L_k^{new} = \varnothing$, k=2,…,K. The objective value of this new set of L_k is bounded above by $TC_{\max}$, the maximum total capacity of any stage. As $TC_{\max} \leq 2A_{\min}$, then the new set has an objective value for **P3**(M) at least as small as the original L_k* set, and from part (i) a stage-spanning bottlenecks never occur if $TC_{\max} \leq 2A_{\min}$.

Note that if one defines an N-stage-spanning bottleneck as one in which there are exactly N non-empty L_k*, then one can adapt the above proof directly to show that such a bottleneck can never occur if $TC_{\max} \leq NA_{\min}$.

Lemma 3:

If a supply chain is $g_{\min}$-type, then (i) a stage-spanning bottleneck can never occur if the total number of products, I, is less than or equal to $2g_{\min}$. Furthermore, if at each stage each individual product is connected to the same total capacity, then (ii) a stage-spanning bottleneck can never occur if the total number of products, I, is less than or equal to $2(g_{\min}+1)$

Proof:

(i) Let $W^k(L_k)$ be the total capacity available to a subset L_k of product at stage k. As each stage k has a g-type configuration with $g_k \geq g_{\min}$, then $W^k(L_k)=0$ iff $L_k=\{\varnothing\}$ and $W^k(L_k) \geq \min\{TC_k, (|L_k|+g_{\min}-1)C_k\}$ iff $L_k \neq \{\varnothing\}$, where as defined earlier TC_k is the total capacity of the stage and $C_k=TC_k/I$. As $TC_{\min}$ is the minimum total stage capacity, $TC_k \geq TC_{\min}$ and $C_k \geq C_{\min}$, therefore $W^k(L_k) \geq \min\{TC_{\min}, (|L_k|+g_{\min}-1)C_{\min}\}$ iff $L_k \neq \{\varnothing\}$. For any subset of products, M, let $\Lambda(M)$ be the set of stages with non-empty L_k*, where L_k* are the minimizing sets in **P3**(M), subject to there being at least two non-empty L_k*. Then $|\Lambda(M)| \geq 2$. For this set of L_k*'s, the objective value for **P3**(M) is given by,

$$\sum_{k=1}^{K} \sum_{j \in P^{k'}(L_k^*)} c_j^k \geq \sum_{k \notin \Lambda(M^*)} W^k(L_k^*) + \sum_{k \in \Lambda(M^*)} W^k(L_k^*) \geq \sum_{k \in \Lambda(M^*)} \min\left\{TC_{\min}, \left(\left|L_k^*\right| + g_{\min} - 1\right)C_{\min}\right\} \tag{L3.1}$$

As the L_k* are non-empty for all $k \in \Lambda(M)$, then $|L_k^*| \geq 1$ for all $k \in \Lambda(M)$. Therefore,

$$\begin{aligned} \sum_{k=1}^{K} \sum_{j \in P^{k'}(L_k^*)} c_j^k &\geq \sum_{k \in \Lambda(M^*)} \min\{TC_{\min}, g_{\min} C_{\min}\} \\ &\geq 2\min\{TC_{\min}, g_{\min} C_{\min}\} \\ &\geq \min\{TC_{\min}, 2g_{\min} C_{\min}\} \end{aligned}$$

Without loss of generality, assume the first stage has the minimum total stage capacity. Construct a new set of L_k, k=1,…,K, as follows. Set $L_1^{new} = \bigcup_{k=1}^{K} L_k^*$ and $L_k^{new} = \varnothing$, k=2,…,K. The objective value of this new set of L_k is bounded above by $TC_{\min}$, the total capacity of stage 1. If $TC_{\min} \leq 2g_{\min}C_{\min}$, or alternatively $I \leq 2g_{\min}$ as $TC_{\min} = IC_{\min}$, then the new set has an objective value for **P3**(M) at least as small as the original L_k* set. Therefore for every possible M, there exists an optimal solution to **P3**(M) with only one non-empty L_k*. Following Lemma 2(a), a stage-spanning bottlenecks can never occur if $I \leq 2g_{\min}$.

Note that if one defines an N-stage-spanning bottleneck as one in which there are exactly N non-empty L_k*, then one can adapt the above proof directly to show that such a bottleneck can never occur if $I \leq Ng_{\min}$.

(ii) If $L_{k'}$* and $L_{k''}$* are non-empty with $L_{k''}$*={i}, i.e. it contains exactly one product, set $L_{k'}^{new} = L_{k'}^* \cup L_{k''}^*$ and $L_{k''}^{new} = \{\varnothing\}$. Then,

$$\sum_{j \in P^{k'}(L_{k'}^{new})} c_j^k \leq \sum_{j \in P^{k'}(L_{k'}^*)} c_j^k + \sum_{j \in P^{k'}(i)} c_j^k = \sum_{j \in P^{k'}(L_{k'}^*)} c_j^k + \sum_{j \in P^{k''}(L_{k''}^*)} c_j^k$$

where the equality occurs because at each stage each individual product is connected to the same total capacity. So, a new set of L_k can be constructed with an optimal value to **P3**(M) at least as small as the original optimum. This is true for any $|L_k^*|=1$. Therefore any optimal set for **P2** with N>1 non-empty L_k* can be transformed into a new optimal set with N-1 non-empty L_k^{new} if some $|L_k^*|=1$. Repeat this process until all non-empty L_k* have $|L_k^*| \geq 2$. So, $|L_k^*| \geq 2$ for any stage-spanning bottleneck. Substituting this into (L3.1) yields a lower bound of $2(g_{\min}+1)C_{\min}$. The rest of proof follows as above.

Note that if one defines an N-stage-spanning bottleneck as one in which there are exactly N non-empty L_k*, then one can adapt the above proof directly to show that such a bottleneck can never occur if $I \leq N(g_{\min}+1)$.

Lemma 4:

If a supply chain is a g_{min}-type and has the following properties,

(i) each stage has a total capacity of at least the total expected demand

(ii) the demands for the I products are independent and identically distributed $N(\mu,\sigma)$

then the probability of any particular LB stage-spanning bottleneck is bounded above by $\Omega_S(I,g_{min})$, where,

$$\Omega(I,g_{\min}) = \Phi\left(\frac{-2(g_{\min}-1)\mu}{\sigma\sqrt{I/2}}\right)^2$$

Proof:

From Section 3.3.1.1, an upper bound on the probability of $(M,L_1,\ldots,L_K)$ being a stage-spanning bottleneck is given by,

$$\Omega_S(M,L_1,\ldots,L_K) = [1-\Phi(z_1)]\Phi(z_2)$$

where

$$z_1 = \frac{\sum_{n=1}^{N}\sum_{j\in P^{k_n}(L_{k_n})} c_j^{k_n} - \sum_{i\in M}\mu_i}{\sqrt{\sum_{i\in M}\sigma_i}} \text{ and } z_2 = \frac{TC_{\min} - \sum_{n=1}^{N}\sum_{P^{k_n}(L_{k_n})} c_j^{k_n} - \sum_{i\notin M}\mu_i}{\sqrt{\sum_{i\notin M}\sigma_i}}$$

and the N stages with non-empty L_k are denoted by $k_1,\ldots,k_N$.

From Lemma 5 below, as each stage in the supply chain has a g-value greater than or equal to g_{min}, then

$$\Omega_S(M,L_1,\ldots,L_K) = [1-\Phi(z_1)]\Phi(z_2) \le [1-\Phi(y_1)]\Phi(y_2) = \Omega_S(x,N,I,g_{\min})$$

where,

$$y_1 = \frac{C_{\min}(x+N(g_{\min}-1)) - \sum_{i\in M}\mu_i}{\sqrt{\sum_{i\in M}\sigma_i}}$$

$$y_2 = \frac{C_{\min}(I-x-N(g_{\min}-1)) - \sum_{i\notin M}\mu_i}{\sqrt{\sum_{i\notin M}\sigma_i}}$$

and x is the number of products in M, i.e. $x=|M|$ and C_{min} is equal to TC_{min}/I, where TC_{min} is the minimum total stage capacity.

As the product demands are iid, then,

$$y_1 = \frac{C_{\min}\left(x + N\left(g_{\min} - 1\right)\right) - x\mu}{\sigma\sqrt{x}} = \frac{N\left(g_{\min} - 1\right)C_{\min} + x(C_{\min} - \mu)}{\sigma\sqrt{x}}$$

$$y_2 = \frac{C_{\min}\left(I - x - N\left(g_{\min} - 1\right)\right)c - (I - x)\mu}{\sigma\sqrt{(I - x)}} = \frac{(I - x)(C_{\min} - \mu) - N\left(g_{\min} - 1\right)C_{\min}}{\sigma\sqrt{(I - x)}}$$

$\Omega_S(x,N,I,g_{\min})=[1-\Phi(y_1)]\Phi(y_2)$ provides an upper bound on $\Omega_S(M,L_1,\ldots,L_K)$, which itself is an upper bound on the probability that $(M,L_1,\ldots,L_K)$ is a stage-spanning bottleneck. Note that this upper bound does not depend on the actual M set and L_k subsets, only on the number of products, x, in M and the number of non-empty L_k subsets, N. As such it is valid for any $(M,L_1,\ldots,L_K)$ for which $|M|=x$ and for which there are N non-empty L_k subsets.

By maximizing $\Omega_S(x,N,I,g_{\min})$ over all possible x for which there can be N non-empty L_k subsets, the dependence of $\Omega_S(x,N,I,g_{\min})$ on x can be removed, to give $\Omega_S(N,I,g_{\min})$, an upper bound on the probability of occurrence of any particular stage-spanning bottleneck with N non-empty L_k subsets (in a supply chain that processes I products and that has a g-value of $g_{\min}$). From Lemma 6 below, x, the number of products in M, must be less than or equal to $I-N(g_{\min}-1)-1$, if M is to be a stage-spanning bottleneck with N non-empty L_k subsets. Therefore,

$$\Omega_S(N, I, g_{\min}) \le \operatorname*{Max}_{x \le I - N\left(g_{\min} - 1\right) - 1} \left\{\Omega_S(x, N, I, g_{\min})\right\} = \operatorname*{Max}_{x \le I - N\left(g_{\min} - 1\right) - 1} \left\{\left[1 - \Phi\left(y_1\right)\right]\Phi\left(y_2\right)\right\}$$

If the minimum total stage capacity equals the expected total demand, then $C_{\min}$ equals the mean product demand, μ, and the maximum occurs at $x^*=I/2$, assuming that $I/2 \ge N(g_{\min}-1)+1$. In this case,

$$\Omega_S(N,I,g_{\min}) = \left[1 - \Phi\left(\frac{N\left(g_{\min} - 1\right)\mu}{\sigma\sqrt{I/2}}\right)\right]\Phi\left(\frac{-N\left(g_{\min} - 1\right)\mu}{\sigma\sqrt{I/2}}\right) = \Phi\left(\frac{-N\left(g_{\min} - 1\right)\mu}{\sigma\sqrt{I/2}}\right)^2 \tag{L4.1}$$

Note that as $\Omega_S(x,N,I,g_{min})$ is decreasing in $C_{\min}$, (L4.1) provides an upper bound on $\Omega_S(N,I,g_{min})$ for any $C_{\min} \ge \mu$ (i.e. any supply chain in which the minimum total stage capacity is greater than or equal to the total expected demand).

$\Omega_S(N,I,g_{min})$ is decreasing in N, i.e. $\Omega_S(2,I,g_{min}) > \Omega_S(N,I,g_{min}) \forall N>2$. By definition a stage-spanning bottleneck must have at least two non-empty L_k subsets, that is $N \ge 2$. Therefore, $\Omega_S(2,I,g_{min})$ gives and upper bound on the probability of occurrence of any particular stage-spanning bottleneck, regardless of the number of non-empty L_k subsets. So, setting $\Omega_S(I,g_{min})=\Omega_S(2,I,g_{min})$ proves the lemma.

Note, if one is interested in the probability of occurrence of an N-stage-spanning bottleneck, then $\Omega_S(N,I,g_{min})$ provides an upper bound on this probability.

Lemma 5:

If each stage in the supply chain has a g-value greater than or equal to $g_{\min}$, then

$$\Omega_S(M,L_1,\ldots,L_K) = [1 - \Phi(z_1)]\Phi(z_2) \le [1 - \Phi(y_1)]\Phi(y_2) = \Omega_S(x,N,I,g_{\min})$$

where,

$$y_1 = \frac{C_{\min}(x + N(g_{\min} - 1)) - \sum_{i \in M} \mu_i}{\sqrt{\sum_{i \in M} \sigma_i}}$$

$$y_2 = \frac{C_{\min}(I - x - N(g_{\min} - 1)) - \sum_{i \notin M} \mu_i}{\sqrt{\sum_{i \notin M} \sigma_i}}$$

and x is the number of products in M, i.e. $x=|M|$. $C_{\min}$ is equal to $TC_{\min}/I$, where $TC_{\min}$ is the minimum total stage capacity.

Proof:

Let $W^k(L_k)$ be the total capacity available to a non-empty subset, L_k, of products at stage k. As each stage k has a g-type configuration with $g_k \ge g_{\min}$, then from equation (11) $W^k(L_k) \ge \min\{TC_k,(|L_k|+g_{\min}-1)C_k\}$. As $TC_{\min}$ is the minimum total stage capacity, $TC_k \ge TC_{\min}$ and $C_k \ge C_{\min}$, therefore $W^k(L_k) \ge \min\{TC_{\min},(|L_k|+g_{\min}-1)C_{\min}\}$. From Lemma 6 below, the number of products in M, must be less than I-$N(g_{\min}$-1), if M is to be a stage-spanning bottleneck with N non-empty L_k subsets. For each non-empty subset L_{kn}, n=1,…,N, $|L_{kn}| \le |M| < I-N(g_{\min}-1)$. Therefore $|L_{kn}|+g_{min}-1<I$ and $(|L_{kn}|+g_{min}-1)C_{\min}<TC_{\min}$, so $\min\{TC_k,(|L_k|+g_{\min}-1)C_k\}=(|L_{kn}|+g_{min}-1)C_{\min}$. Therefore, $W^k(L_{kn}) \ge (|L_{kn}|+g_{\min}-1)C_{\min}$ for n=1,…,N. So,

$$\sum_{n=1}^{N} \sum_{j \in P^{k_n}(L_{k_n})} c_j^{k_n} = \sum_{n=1}^{N} W^{k_n}(L_{k_n}) \ge \sum_{n=1}^{N} \left[\left(\left|L_{k_n}\right| + g_{\min} - 1\right)C_{\min}\right] = \left(|M| + N(g_{\min} - 1)\right)C_{\min}$$

Let $|M|=x$. Therefore

$$z_1 = \frac{\sum_{n=1}^{N} \sum_{j \in P^{k_n}(L_{k_n})} c_j^{k_n} - \sum_{i \in M} \mu_i}{\sqrt{\sum_{i \in M} \sigma_i}} \ge \frac{C_{\min}(x + N(g_{\min} - 1)) - \sum_{i \in M} \mu_i}{\sqrt{\sum_{i \in M} \sigma_i}} = y_1$$

$$z_2 = \frac{TC_{\min} - \sum_{n=1}^{N} \sum_{P^{k_n}(L_{k_n})} c_j^{k_n} - \sum_{i \notin M^*} \mu_i}{\sqrt{\sum_{i \notin M} \sigma_i}} \leq \frac{TC_{\min} - C_{\min}\left(x + N\left(g_{\min} - 1\right)\right) - \sum_{i \notin M} \mu_i}{\sqrt{\sum_{i \notin M^{**}} \sigma_i}}$$

$$= \frac{C_{\min}\left(I - x - N\left(g_{\min} - 1\right)\right) - \sum_{i \notin M} \mu_i}{\sqrt{\sum_{i \notin M} \sigma_i}} = y_2$$

as $TC_{\min}=IC_{\min}$. Now,

$$\Omega_S(M, L_1, \ldots, L_K) = [1 - \Phi(z_1)]\Phi(z_2) \leq [1 - \Phi(y_1)]\Phi(y_2) = \Omega_S(x, N, I, g_{\min})$$

as $y_1 \leq z_1$, $y_2 \geq z_2$ and $\Phi(z)$ is increasing in z.

Lemma 6:

If a supply chain is $g_{\min}$-type, then a stage-spanning bottleneck with N non-empty L_k* and $|M^*| \geq I - N(g_{\min}-1)$ can never occur.

Proof:

Let $(M^*, L_1^*, \ldots, L_K^*)$ be an LB stage-spanning bottleneck with N non-empty L_k* and $|M^*| > I - N(h-1)$. Let $W^k(L_k)$ be the total capacity available to a subset L_k of product at stage k. As each stage k has a g-type configuration with $g_k \geq g_{\min}$, then $W^k(L_k)=0$ iff $L_k=\{\emptyset\}$ and $W^k(L_k) \geq \min\{TC_k, (|L_k|+g_{\min}-1)C_k\}$ iff $L_k \neq \{\emptyset\}$, where as defined earlier TC_k is the total capacity of the stage and $C_k = TC_k/I$. As $TC_{\min}$ is the minimum total stage capacity, $TC_k \geq TC_{\min}$ and $C_k \geq C_{\min}$, therefore $W^k(L_k) \geq \min\{TC_{\min}, (|L_k|+g_{\min}-1)C_{\min}\}$ iff $L_k \neq \{\emptyset\}$. Let $\Lambda(M^*)$ be the set of stages with non-empty L_k*. The objective value for **P3**(M*) is,

$$\sum_{k=1}^{K} \sum_{j \in P^{k'}(L_k^*)} c_j^k \geq \sum_{k \notin \Lambda(M^*)} W^k(L_k^*) + \sum_{k \in \Lambda(M^*)} W^k(L_k^*)$$

$$\geq \sum_{k \in \Lambda(M^*)} \min\left\{TC_{\min}, \left(\left|L_k^*\right| + g_{\min} - 1\right)C_{\min}\right\}$$

$$\geq \min\left\{TC_{\min}, \sum_{k \in \Lambda(M^*)} \left[\left(\left|L_k^*\right| + g_{\min} - 1\right)C_{\min}\right]\right\}$$

$$= \min\left\{TC_{\min}, \left(\left|M^*\right| + N\left(g_{\min} - 1\right)\right)C_{\min}\right\}$$

Without loss of generality, assume the first stage has the minimum total stage capacity. Construct a new set of L_k, k=1,..K, as follows. Set $L_1^{new} = \bigcup_{k=1}^{K} L_k^*$ and $L_k^{new} = \varnothing$, k=2,…,K. The objective value of this new set of L_k is bounded above by $TC_{\min}$, the total capacity of stage 1. If $TC_{\min} \le (|M^*|+N(g_{\min}-1))C_{\min}$, then the new set has an objective value for **P3**(M) at least as small as the original L_k* set. Therefore for every possible M, there exists an optimal solution to **P3**(M) with only one non-empty L_k*. Following the addendum to Lemma 2(i) regarding N-stage-spanning bottlenecks, if $TC_{\min} \le (|M^*|+N(g_{\min}-1))C_{\min}$, then a stage-spanning bottlenecks with N non-empty L_k* can never occur. $TC_{\min}=IC_{\min}$ and therefore a stage-spanning bottlenecks with N non-empty L_k* can never occur if $I \le |M^*|+N(g_{\min}-1)$.

Lemma 7:

For a two-stage 4-product supply chain with each stage having 4 plants and a type h=2 chain configuration, and all plant capacities being equal (=c), if the product demands are iid $N(\mu,\sigma)$, then the probability that the stand alone shortfalls for the two stages are the same is greater than or equal to,

$$1-\left(8\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\left[\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)+\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\right]+8\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)\right)$$

Proof:

Let SF_k denote the random variable for the shortfall for stage k, k=1,2. For a given demand realization $d_1,\ldots,d_I$, let m_k* be the maximizing set for,

$$\max_{m_k}\{\sum_{i\in m_k} d_i - \sum_{j\in P^k(m_k)} c_j^k\}$$
$$\text{subject to } m \subseteq \{1,\ldots,I\}$$

and let sf_k be the optimal value for stage k, k=1,2 (i.e. the shortfall for this realization). Let the indicator function I_{12} be such that I_{12}=0 if m_1*=m_2 and I_{12}=1 if m_1*≠m_2. As $c^1_j=c^2_j$, j=1,…,J, $sf_1 \ne sf_2$ only if $m_1^* \ne m_2^*$. Therefore P[$SF_1 \ne SF_2$]≤P[I_{12}=1]. Because the demand distribution is continuous the probability that $M_1^* \ne M_2^*$ and $SF_1=SF_2$ is zero. Therefore in this case, P[$SF_1 \ne SF_2$]=[I_{12}=1]. Note that the upper bound in the lemma is still valid if P[$SF_1 \ne SF_2$]≤P[I_{12}=1].

The only possible chains for a 4-product 4-plant stage are {1,2,3,4}, {1,2,4,3} and {1,3,2,4}. Let stage 1 have a {1,2,3,4} chain and stage 2 have a {1,2,4,3} chain. For any demand realization, the possible m_1* sets are {∅},{1},{2},{3},{4},{1,2},{2,3}, {3,4},{4,1},{1,2,3,4} and the possible m_2* sets are {∅},{1},{2},{3},{4},{1,2}, {2,4},{3,4},{1,3},{1,2,3,4}. The events in which I_{12}=1 ($m_1^* \neq m_2^*$) can be partitioned into the following mutually exclusive (and exhaustive) events:

Event 1: $m_1^*=\{\varnothing\} \cap m_2^* \neq \{\varnothing\}$

Event 2: $m_1^* \neq \{\varnothing\} \cap m_2^*=\{\varnothing\}$

Event 3: $m_1^*=\{1,2,3,4\} \cap m_2^* \neq \{\varnothing\} \cap m_2^* \neq \{1,2,3,4\}$

Event 4: $m_1^* \neq \{\varnothing\} \cap m_1^* \neq \{1,2,3,4\} \cap m_2^*=\{1,2,3,4\}$

Event 5: $m_1^*=\{i\} \cap m_2^* \neq \{i\} \cap m_2^* \neq \{1,2,3,4\} \cap m_2^* \neq \{\varnothing\}$ i=1,2,3,4

Event 6: $(m_1^*=\{1,2,3,4\} \cup m_1^*=\{\varnothing\}) \cap m_2^*=\{i\}$ i=1,2,3,4

Event 7: (a) $m_1^*=\{1,2\} \cap (m_2^* \in \{1,3\} \cup \{3,4\} \cup \{2,4\})$

(b) $m_1^*=\{2,3\} \cap (m_2^* \in \{1,3\} \cup \{3,4\} \cup \{2,4\} \cup \{1,2\})$

(c) $m_1^*=\{3,4\} \cap (m_2^* \in \{1,3\} \cup \{2,4\} \cup \{1,2\})$

(d) $m_1^*=\{4,1\} \cap (m_2^* \in \{1,3\} \cup \{3,4\} \cup \{2,4\} \cup \{1,2\})$

I now develop upper-bounds for the probability of each event.

Event 1: $\boldsymbol{m_1^*=\{\varnothing\}, m_2^* \neq \{\varnothing\}}$

$m_1^*=\{\varnothing\}$ implies:

$$d_i - 2c \le 0 \qquad \text{i} = 1,2,3,4$$
$$d_i + d_{i+1} - 3c \le 0 \qquad \text{i} = 1,2,3,4$$
$$d_1 + d_2 + d_3 + d_4 - 4c \le 0$$

where i+1=1 if i =4. Therefore, the only possible m_2* (≠{∅}) sets are {1,3} or {2,4}. For $m_1^*=\{\varnothing\}$ and $m_2^*=\{1,3\}$, the following four conditions are necessary (but not sufficient),

$$\text{(i) } d_i - 2c \le 0 \qquad \text{i} = 1,2,3,4$$
$$\text{(ii) } d_i + d_{i+1} - 3c \le 0 \qquad \text{i} = 1,2,3,4$$
$$\text{(iii) } d_1 + d_3 - 3c > 0$$
$$\text{(iv) } d_1 + d_2 + d_3 + d_4 - 4c \le 0$$

(iii) and (iv) imply that the following is a necessary condition for $m_1^*=\{\varnothing\}$ and $m_2^*=\{1,3\}$,

$$d_1 + d_3 > 3c \quad \text{and} \quad d_2 + d_4 \le c$$

As the demands are iid $N(\mu,\sigma)$, the probability of this event is given by,

$$\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)$$

and this is an upper-bound on P[m_1*={∅} and m_2*={1,3}]. As above, it can also be shown to be an upper-bound on P[m_1*={∅} and m_2*={2,4}]. Therefore,

$$\mathrm{P[Event\,1]} \leq 2\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)$$

Event 2: $\boldsymbol{m_1^*\neq\{\varnothing\}\cap m_2^*=\{\varnothing\}}$

Using the same derivation as for Event 1, but adapting for Event 2, the same upper-bound as 1 can be developed. Therefore,

$$\mathrm{P[Event\ 2]} \leq 2\Phi\left(\frac{2\mu-3C}{\sqrt{2}\sigma}\right)\Phi\left(\frac{C-2\mu}{\sqrt{2}\sigma}\right)$$

Event 3: $\boldsymbol{m_1^*=\{1,2,3,4\}\cap m_2^*\neq\{\varnothing\}\cap m_2^*\neq\{1,2,3,4\}}$

m_1*={1,2,3,4} implies:

$$d_1+d_2+d_3+d_4-4C>0$$
$$d_1+d_2+d_3+d_4-4C>d_i-2c \qquad \mathrm{i}=1,2,3,4$$
$$d_1+d_2+d_3+d_4-4C>d_i+d_{i+1}-3c \qquad \mathrm{i}=1,2,3,4$$

Therefore, the only possible m_2* sets (≠{∅},≠{1,2,3,4}) or are {1,3} or {2,4}.

For m_1*={1,2,3,4} and m_2*={1,3}, the following two conditions are necessary (but not sufficient),

(i) $d_1+d_3-3c>0$
(ii) $d_1+d_3-3c\geq d_1+d_2+d_3+d_4-4c \Rightarrow d_2+d_4-c\leq 0$

As the demands are iid $N(\mu,\sigma)$, the probability of this event is given by,

$$\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)$$

and this is an upper-bound on P[m_1*={1,2,3,4} and m_2*={1,3}]. As above, it can also be shown to be an upper-bound on P[m_1*={1,2,3,4} and m_2*={2,4}]. Therefore,

$$\mathrm{P[Event\,3]} \leq 2\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)$$

Event 4: $\boldsymbol{m_1*\neq\{\varnothing\}\cap m_1*\neq\{1,2,3,4\}\cap m_2*=\{1,2,3,4\}}$

Using the same derivation as for Event 3, but adapting for Event 4, the same upper-bound as 1 can be developed. Therefore,

$$\text{P[Event 4]} \leq 2\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)$$

Event 5: $\boldsymbol{m_1*=\{i\}\cap m_2*\neq\{i\}\cap m_2*\neq\{1,2,3,4\}\cap m_2*\neq\{\varnothing\}}$ **i=1,2,3,4**

m_1*={1} implies

$$d_1 - 2c \geq 0$$
$$d_i - 2c \leq d_1 - 2c \qquad \text{i} = 2,3,4$$
$$d_i + d_{i+1} - 3c \leq d_1 - 2c \qquad \text{i} = 1,2,3,4$$
$$d_1 + d_2 + d_3 + d_4 - 4c \leq d_1 - 2c$$

Canceling terms, then

(i) $d_1 - 2c \geq 0$

(ii) $d_i \leq d_1$ $\qquad$ i = 2,3,4

(iii)(a) $d_2 - c \leq 0$ (b) $d_2 + d_3 - d_1 - c \leq 0$ (c) $d_3 + d_4 - d_1 - c \leq 0$ (d) $d_4 - c \leq 0$

(iv) $d_2 + d_3 + d_4 - 2c \leq 0$

Therefore, the only possible m_2* set (≠{1},{1,2,3,4},{∅}) is {1,3}. m_2* cannot be ={j}, j≠1, from (ii). m_2* cannot be {2,4} from (iv). For m_1*={1} and m_2*={1,3}, the following conditions are necessary (but not sufficient),

(v) $d_1 - 2c \geq 0$

(vi) $d_1 + d_3 - 3c \geq d_1 + d_2 + d_3 + d_4 - 4c \Rightarrow d_2 + d_4 - c \leq 0$

(vii) $d_1 + d_3 - 3c \geq d_1 - 2c \Rightarrow d_3 - c \geq 0$

As the demands are iid $N(\mu,\sigma)$, the probability of this event is given by,

$$\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)$$

and this is an upper-bound on P[m_1*={1} and m_2*≠{1},{1,2,3,4},{∅})]. The same upper-bound can be developed for P[m_1*={i} and m_2*≠{i},{1,2,3,4},{∅})], i=2,3,4. Therefore,

$$\text{P[Event 5]} \leq 4\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)$$

Event 6: $(m_1^*=\{1,2,3,4\}\cup m_1^*=\{\varnothing\})\cap m_2^*=\{i\}$ **i=1,2,3,4**

Using the same derivation as for Event 3, but adapting for Event 4, the same upper-bound as 1 can be developed. Therefore,

$$\text{P[Event 6]} \le 4\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)$$

Event 7: **(a)** $m_1^*=\{1,2\}\cap(m_2^*\in\{1,3\}\cup\{3,4\}\cup\{2,4\})$

(b) $m_1^*=\{2,3\}\cap(m_2^*\in\{1,3\}\cup\{3,4\}\cup\{2,4\}\cup\{2,1\})$

(c) $m_1^*=\{3,4\}\cap(m_2^*\in\{1,3\}\cup\{2,4\}\cup\{2,1\})$

(d) $m_1^*=\{4,1\}\cap(m_2^*\in\{1,3\}\cup\{3,4\}\cup\{2,4\}\cup\{2,1\})$

(a) $m_1^*=\{1,2\}$ implies

$$d_1+d_2-3c\ge 0$$
$$d_i-2c\le d_1+d_2-3c \qquad \text{i} = 2,3,4$$
$$d_i+d_{i+1}-3c\le d_1+d_2-3c \qquad \text{i} = 2,3,4$$
$$d_1+d_2+d_3+d_4-4c\le d_1+d_2-3c$$

Therefore $m_2^*\neq\{3,4\}$. Only need to consider $m_2^*=\{1,3\}$ or $\{2,4\}$.

<u>$m_1^*=\{1,2\}$ $m_2^*=\{1,3\}$:</u>

Canceling terms in the above equations, then

(i) $d_1+d_2-3c\ge 0$

(ii) (a) $d_2-c\ge 0$ (b) $d_1-c\ge 0$ (c) $d_1+d_2-d_3-c\ge 0$ (c) $d_1+d_2-d_4-c\ge 0$

(iii) (a) $d_1-d_3\ge 0$ (b) $d_1+d_2-d_3-d_4\ge 0$ (c) $d_2-d_4\ge 0$

(iv) $d_3+d_4-c\le 0$

But, $m_2^*=\{1,3\}$ implies

(v) $d_1+d_2+d_3+d_4-4c\le d_1+d_3-3c\Rightarrow d_2+d_4-c\le 0$

Using (iv) and (v), then $d_2+d_3+2d_4-2c\le 0$. From (ii)(a) $d_2\ge c$. Therefore,

(vi) $c+d_3+2d_4-2c\le 0\Rightarrow d_3+2d_4-c\le 0$.

(i) and (vi) are thus necessary conditions for $m_1^*=\{1,2\}$ and $m_1^*=\{1,3\}$. As the demands are iid $N(\mu,\sigma)$, the probability of this event is given by,

$$\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)$$

This is an upper-bound on the probability that m_1*={1,2} m_2*={1,3}. It can also be shown to be an upper-bound on m_1*={1,2} m_2*={2,4}. Therefore,

$$\text{P[Event 7(a)]} \le 2\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)$$

Events 7(b),(c) and (d) have similar upper bounds. Therefore,

$$\text{P[Event 7]} \le 8\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)$$

The probability of I_{12}=1 ($m_1^* \neq m_2^*$) is the sum of the probability of the above events. Therefore,

$\text{P}[I_{12}] \le \sum_{\text{n}=1}^{7} \text{P[Event n]}$. So,

$$\text{P}[I_{12}] \le 8\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\left[\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)+\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\right]+8\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)$$

As, $\text{P}[SF_1 \neq SF_2] \le \text{P}[I_{12}=1]$,

$$\text{P}[SF_1 \neq SF_2] \le 8\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\left[\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)+\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\right]+8\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)$$

and so,

$$\text{P}[SF_1 = SF_2] \ge 1 - \left(8\Phi\left(\frac{2\mu-3c}{\sqrt{2}\sigma}\right)\left[\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)+\Phi\left(\frac{2c-3\mu}{\sqrt{5}\sigma}\right)\right]+8\Phi\left(\frac{\mu-2c}{\sigma}\right)\Phi\left(\frac{c-2\mu}{\sqrt{2}\sigma}\right)\Phi\left(\frac{\mu-c}{\sigma}\right)\right)$$

This proof can be repeated for the other possible chain pairings.

5.3 Appendix 3

This appendix contains proofs of the lemmas from Chapter 4 and an algorithm for generating the set of possible stage subsets over which the maximization in Lemma 1 is evaluated (c.f. Lemma 3).

Lemma 1:

(i) A lower bound on the minimum total shortfall in problem **P3** is given by

$$\max_{\Lambda \subseteq \{1,\ldots,K\}} \{ \sum_{f \in P(\Lambda)} d_f - \sum_{k \in \Lambda} c_k \}$$

(ii) If the path-stage matrix **B** is totally unimodular, then the minimum total shortfall in problem **P3** equals the lower bound in (i).

Proof:

Problem **P3** is given by the linear program,

$$\text{Min} \{ \sum_{f=1}^{f} sf_f \}$$

subject to

$$1.\ y_f + sf_f \geq d_f \qquad f \in F$$

$$2.\ \sum_{f \in P(k)} y_f \leq c_k \qquad k = 1,\ldots,K$$

$$3.\ \mathbf{y}, \mathbf{sf} \geq \mathbf{0}$$

Let π_f be the dual variable for the Type 1 constraints and μ_k be the dual variable for the Type 2 constraints. Letting ν_k=-μ_k gives us the following dual formulation (**D3**),

$$\underset{\boldsymbol{\pi},\boldsymbol{\nu}}{\text{Max}} \{ \sum_{f \in F} \pi_f d_f - \sum_{k=1}^{K} \nu_k c_k \}$$

subject to

$$1.\qquad \pi_f \leq 1 \qquad f \in F$$

$$2.\qquad \pi_f \leq \sum_{k \in Q(f)} \nu_k \qquad f \in F$$

$$3.\qquad \boldsymbol{\pi} \geq \mathbf{0} \quad , \quad \boldsymbol{\nu} \geq \mathbf{0}$$

Remember that $P(\Lambda)$ is the set of flow paths that are processed by at least one stage $k \in \Lambda$ and $Q(f)$ is the set of stages that process flow path f.

Let problem **D4** be the same as **D3** but with the additional constraint that the solution be binary. For a feasible solution $(\boldsymbol{\pi}, \mathbf{v})$, let Λ be the subset of stages k=1,…,K for which $\nu_k = 1$. This solution can be optimal only if $\pi_f = 1$ $\forall f \in P(\Lambda)$ and $\pi_f = 0$ $\forall f \notin P(\Lambda)$, where the second condition is required for feasibility. To see this, consider a solution to **D4** in which $\pi_f = 0$ for some flow path $f \in P(\Lambda)$. This solution can be improved upon by setting $\pi_f = 1$. This is a feasible solution with an increased objective function. Each of the possible optimal solutions is therefore completely specified by the subset Λ. The objective value for such a solution is given by,

$$\sum_{f \in P(\Lambda)} d_f - \sum_{k \in \Lambda} c_k$$

Any subset Λ of stages 1,…,K is a possible candidate for optimality and therefore the optimum objective value to **D4** is given by

$$\max_{\Lambda \subseteq \{1,\ldots,K\}} \{ \sum_{f \in P(\Lambda)} d_f - \sum_{k \in \Lambda} c_k \}$$

As all solutions to **D4** are feasible for **D3**, the optimal objective for **D4** is a lower bound on the optimal objective value to **D3**. From duality the minimum shortfall for **P3** is equal to the optimal objective value to **D3**.

(ii) The constraint matrix, **A**, for **D3** is given by,

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{B} \end{bmatrix}$$

where **B** is the path-stage matrix. The path-stage matrix is the matrix in which there is a row for each flow path, a column for each stage k and b_{fk}=1 if $k \in Q(f)$ and b_{fk}=0 if $k \notin Q(f)$. In other words, element (f,k) is 1 if flow path f requires stage k and 0 otherwise.

A is totally unimodular (TU) if **B** is TU. This follows from the fact that total unimodularity is preserved under the following operations (Schrijver, 1987)

(a) multiplying a column by –1

(b) adding a row or column with at most one nonzero, being +/-1.

If **B** is TU then **–B** is TU using (a). Then **[I –B]** is TU using (b) and **A** is TU using (b) again.

So, if the path-stage matrix **B** is TU, then the constraint matrix **A** for **D3** is TU and therefore optimal solution to **D3** is integral. The Type 1 constraints ensure that $\pi_f \leq 1 \; f \in F$. No optimal solution can have any $\nu_k > 1$ as the objective function can be decreased by setting such a $\nu_k = 1$

while still maintaining feasibility. The optimal solution to **D3** is therefore binary and thus from part (i), the optimal objective value for **D3** and **P3** is given by,

$$\max_{\Lambda \subseteq \{1,\ldots,K\}} \{ \sum_{f \in P(\Lambda)} d_f - \sum_{k \in \Lambda} c_k \}$$

Lemma 2:

The path-stage matrices for the Alcalde Job Shop and for Work Center A are both totally unimodular (TU).

Proof:

The path-stage matrix for the Alcalde job shop $\mathbf{B}_{Alcalde}$ is given by,

$$\mathbf{B}_{Alcalde} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Total unimodularity is preserved under addition of a column with at most one nonzero, being +/-1 (Schrijver, 1987). Therefore $\mathbf{B}_{Alcalde}$ is TU if the following submatrix **S** of $\mathbf{B}_{Alcalde}$ is TU.

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

S is TU if each collection of columns of **S** can be split into two parts so that the sum of the columns in one part minus the sum of columns in the other part is a vector with entries only 0, +1, and –1 (Schrijver, 1987).

Clearly any collection of two columns of **S** can be split into two such parts as each column contains only 0's or +1's. It only remains to show that a collection of all three columns can be split in this manner. Assign the first and second column to one part and the third column to the other part. The sum of columns in the first part is a vector with entries only +1 and +2. The sum of columns in the second part is a vector with every entry equal to +1. Therefore the sum of columns in one part minus the sum of columns in the other part is a vector with entries only 0, +1, and –1. **S** is TU and thus $\mathbf{B}_{Alcalde}$ is TU.

The path-stage matrix for Work Center A $\mathbf{B}_{WCA}$ is given by,

$$\mathbf{B}_{WCA} = \left[\begin{array}{c|cccccccccccc} & A & B & C & D & E & F & G & I & K & L & N & P \\ \hline BP & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ ABP & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ ABFP & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ ACP & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ ACFP & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ ADFP & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ ADGP & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ ADIP & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ ADKP & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ ADLP & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ ADNP & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ ADP & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ AEP & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

where I have added the stages and flowpaths for clarity. As discussed in Section 4.3.1, stages *I* and *N* refer to the aggregated stages *IJ* and *NO*. Column vectors are denoted by the associated stage letter, e.g. *A* or *P*.

Total unimodularity is preserved under both column permutation and the addition of a column with at most one nonzero, being +/-1 (Schrijver, 1987). Therefore $\mathbf{B}_{WCA}$ is TU if the following submatrix **S** of $\mathbf{B}_{WCA}$ is TU.

$$\mathbf{S} = \begin{bmatrix} P & A & F & B & C & D \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the columns *G*-*N* have been removed and the column order has been rearranged. The matrix **S** has the following two properties.

[1] Any collection of columns from {P,A,F} can be split into two parts so that the sum of columns in the first part minus the sum of columns in the second part is a vector with entries only 0 or +1. To see this *P*-*A*, *P*-*F* and *P*+*F*-*A* are all vectors with entries only 0 or +1.

[2] The sum of any collection of columns from {B,C,D} is a vector containing only 0 or +1 entries.

From [1] and [2], any collection of columns of **S** can be split into two parts so that the sum of the columns in one part minus the sum of columns in the other part is a vector with entries only 0, +1, and –1. Therefore **S** is TU (Schrijver, 1987) and thus $\mathbf{B}_{WCA}$ is TU.

Lemma 3:

Any subset Λ that can be partitioned into two disjoint subsets Λ_m and Λ_n such that $P(\Lambda_m) \subseteq P(\Lambda_n)$, can be omitted from the set of subsets over which the maximum in Lemma 1 is evaluated.

Proof:

If Λ can be partitioned into two disjoint subsets Λ_m and Λ_n such that $P(\Lambda_m)\subseteq P(\Lambda_n)$, then $P(\Lambda)=P(\Lambda_n)$. This implies

$$\sum_{f\in P(\Lambda)} d_f - \sum_{k\in\Lambda} c_k = \sum_{f\in P(\Lambda_n)} d_f - \sum_{k\in\Lambda_m} c_k - \sum_{k\in\Lambda_n} c_k \le \sum_{f\in P(\Lambda_n)} d_f - \sum_{k\in\Lambda_n} c_k$$

and therefore Λ cannot be the optimum in the above maximization.

Algorithm for generating the set *L* of possible stage combinations in the maximization of Lemma 1.

The set of stage combinations (or subsets) over which the maximum in Lemma 1 is evaluated does not contain every single possible combination of stages. Lemma 3 states that some stage combinations (or subsets) can be removed. An equivalent statement to Lemma 3 is given by,

If $P(\Lambda_m)\subseteq P(\Lambda_n)$, then $\Lambda=\Lambda_m\cup\Lambda_n$ can be omitted from the set of subsets over which the maximum in Lemma 1 is evaluated.

Denote the set of possible subsets by the set *L*. By using this version of Lemma 3, it is possible to compare two subsets to see whether the union of the two subsets can is contained in *L*. Remember a subset Λ corresponds to a combination of stages and P(Λ) corresponds to the set of flow paths processed by any stage in Λ.

This comparison can be done using matrix algebra as follows. Let $P(\Lambda_m)$ be specified by a row vector

$$\mathbf{R^m} = \left[r_1^m, \cdots, r_F^m\right]$$

where the element r_f^m=1if $f\in P(\Lambda_m)$ and equals 0 otherwise for $f = 1,\ldots,F$. In other words, if a flow path is processed by some stage in Λ_m, then the corresponding flow path entry equals one. Let $\mathbf{R^n}$ be the flow path row vector for Λ_n. Let $\mathbf{R^{n-m}} = \mathbf{R^n} - \mathbf{R^m}$. If $\mathbf{R^{n-m}} \ge \mathbf{0}$, i.e. each element is non-negative, then $P(\Lambda_m)\subseteq P(\Lambda_n)$. Likewise if $\mathbf{R^{m-n}} \ge \mathbf{0}$, then $P(\Lambda_n)\subseteq P(\Lambda_n)$. An algorithm for comparing two subsets to see if the union of the subsets can be one of the subsets in L can be specified as follows.

ALGORITHM: COMPARE($\mathbf{R^n}, \mathbf{R^m}$)

$\mathbf{R^{n-m}} = \mathbf{R^n} - \mathbf{R^m}$

$\mathbf{R^{m-n}} = \mathbf{R^m} - \mathbf{R^n}$

IF $\mathbf{R^{n-m}} \geq \mathbf{0}$

    RETURN FALSE

ELSE IF $\mathbf{R^{m-n}} \geq \mathbf{0}$

    RETURN FALSE

ELSE

    RETURN TRUE

END

The *COMPARE*($\mathbf{R^n}, \mathbf{R^m}$) algorithm determines whether the union of two subsets is contained in the set *L*. However, we still need to generate the set *L*. The algorithm for doing this is called *GENERATE L.*

This algorithm works as follows. The set *L* starts as an empty set. All subsets containing only one stage are then added to *L*. Next, the possible subsets containing two stages are added to L. Then the possible subsets with three stages are added, then four stages, etc. up until *K* stages. At this stage all possible subsets have been evaluated. If a set with *j*-1 stages, say Λ_{j-1}, is not in *L*, then no set that contains Λ_{j-1} will be in *L*. Therefore, when evaluating the subsets containing *j* stages, one only needs to consider the subsets that are formed by the union of a single stage and a subset containing *j*-1 stages that is already in *L*.

The algorithm is an iterative algorithm. Each iteration corresponds to the evaluation of the subsets containing *j* stages. At the start of the iteration, the subsets in *L* that contain *j*-1 stages will have been identified. Each of these "*j*-1" subsets will have a row vector corresponding to the flow paths processed by any stage in this subset (See above). These row vectors will form the matrix $\mathbf{M^{j-1}}$. The number of *j*-1 subsets will be specified by N^{j-1}. The n^{th} row vector of $\mathbf{M^{j-1}}$ will be identified by $\left[\mathbf{M^{j-1}}\right]_n$ .In the algorithm, the stage subset corresponding to the n^{th} row vector of $\mathbf{M^{j-1}}$ will be identified by *stage_subset*$\left[\mathbf{M^{j-1}}\right]_n$. In general, for any set Λ $\mathbf{R}$\{Λ\} is the row vector corresponding to the flow path subset P(Λ).

ALGORITHM: GENERATE L

Initialization: Adding the single stage subsets to L

L=EMPTY SET

FOR k=1 TO K

INCLUDE {k} in L

ATTACH row vector $\mathbf{R}\{k\}$ to bottom of matrix $\mathbf{M}^{\mathbf{1}}$

END k LOOP

Iteration: Generating the subsets with j stages that are in L, j=2,…,K

FOR j=2 TO K

FOR k=1 TO K

FOR n=1 TO N^{j-1}

COMPARE $(\mathbf{R}\{k\}, [\mathbf{M}^{\mathbf{j-1}}]_n)$

IF RETURN = TRUE

$\{newset\}=\{k\}\cup\{\ stage\_subset[\mathbf{M}^{\mathbf{j-1}}]_n\ \}$

INCLUDE {*newset*} in L

ATTACH row vector $\mathbf{R}\{newset\}$ to bottom of matrix $\mathbf{M}^{\mathbf{1}}$

END n LOOP

END k LOOP

END j LOOP

END